Coursework 2 - Quantum mechanics

Deadline: 28 February 2008

1. Let us consider a one-dimensional harmonic oscillator. At an initial time $t = 0$, this system is prepared in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle + \frac{1}{\sqrt{3}}|3\rangle,$$

which is a linear superposition of the first, second and third excited eigenstates of its Hamiltonian $\hat{H} = \hat{T} + \hat{V}$, where $\hat{T}$ and $\hat{V}$,denote the kinetic and potential energy of this system, respectively.

   (a) Compute the initial expectation value $\langle\psi(0)|\hat{T}|\psi(0)\rangle$ of the kinetic energy.

   (b) As the system evolves in time, the expectation value $\langle\psi(t)|\hat{T}|\psi(t)\rangle$ changes, with $|\psi(t)\rangle = U(t,0)|\psi(0)\rangle$ ($U(t,0)$ is the time evolution operator). Obtain explicit expressions for this time dependent expectation value.
   Hint: Use the lowering and raising operators, together with the fact that

$$\begin{aligned} \hat{a}_+|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ \hat{a}_-|n\rangle &= \sqrt{n}\,|n-1\rangle. \end{aligned}$$

2. Consider the spherical harmonics

$$\begin{aligned} Y_1^1(\theta,\phi) &= -\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{i\phi} \\ Y_1^0(\theta,\phi) &= \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \\ Y_1^{-1}(\theta,\phi) &= \left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{-i\phi}. \end{aligned}$$

   (a) Which of the above-stated functions is an eigenfunction of $\hat{L}_z$ with eigenvalue $+\hbar$?Why?

   (b) Write the spherical harmonics $\tilde{Y}_1^1(\theta,\phi), \tilde{Y}_1^0(\theta,\phi)$ and $\tilde{Y}_1^{-1}(\theta,\phi)$ obtained by performing a clockwise 90^0 rotation of $Y_1^1(\theta,\phi), Y_1^0(\theta,\phi)$ and $Y_1^{-1}(\theta,\phi)$ about the x axis. Show that the functions obtained are eigenfunctions of $\hat{L}_y$ with the eigenvalues $+\hbar, 0, -\hbar$, respectively.
   Hint: write the spherical harmonics $Y_l^m(\theta,\phi)$ in terms of the cartesian coordinates x, y, z and of $r = \sqrt{x^2+y^2+z^2}$ and consider the

effect of rotating these functions as described above (how would the coordinates change?). Subsequently, transform back to the polar spherical coordinates.

(c) Can the rotated spherical harmonics be written as a superposition of eigenfunctions of $\hat{L}_z$? If so, how are they explicitly written?Which eigenvalues of $\hat{L}_z$ can be measured in each case?
Hint: The task is easier to perform if you write x/r, y/r and z/r in terms of spherical harmonics.

3. A system is in a state in which the z component of the orbital angular momentum L_z has the precise value $\hbar$. Consider now the operators $\hat{L}_1 = 3\hat{L}_x + \hat{L}_y$ and $\hat{L}_2 = 2\hat{L}_y$.

   (a) What are the uncertainties of these operators in the given state?

   (b) In the given state, may the square of the angular momentum have a precise value?Justify your answer.
   Hint: Throughout, employ the commutation relations involving the components of the angular momentum operator and the generalized uncertainty relation for two observables A,B seen in class.