Coursework 1 - Quantum mechanics
Deadline: 8th of February 2008

1. The electron in a Hydrogen atom is prepared in the state described by the wavefunction
$$\psi(r) = c_1\psi_{100}(r) + c_2\psi_{200}(r),$$
with
$$\psi_{100}(r) = \frac{1}{\sqrt{\pi a_0^3}} \exp[-r/a_0]$$
and
$$\psi_{200}(r) = \frac{1}{\sqrt{8\pi a_0^3}} \left(1 - \frac{r}{2a_0}\right) \exp[-r/(2a_0)].$$
Thereby, both $\psi_{100}(r)$ and $\psi_{200}(r)$ are normalized wavefunctions.

   (a) Show that, indeed, $\psi_{200}(r)$ is correctly normalized.

   (b) What is the relationship between c_1 and c_2 so that the wavefunction is correctly normalized?

   (c) Compute the expectation value of the electron potential energy
   $$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}.$$
   Consider equal contributions from each term in $\psi(r)$ (i.e., $c_1 = c_2$).

   (d) What are the possible values of $V(r)$ that can be measured? Would you expect this from any of the postulates of quantum mechanics? Discuss

   In the above-stated exercises, you will need to empoly the radial integrals
   $$I_n = \int_0^\infty r^n \exp[-\alpha r] dr = \frac{n!}{\alpha^{n+1}}.$$

2. Let $\mathcal{X}$ be the space of the differentiable functions $\psi(x)$, which are defined within an interval x $\in [a,b]$, with $\psi(a) = \psi(b) = 0$. Prove that

   (a) the translation operator $T(\Delta)$, which is defined by
   $$T(\Delta)\psi(x) = \psi(x + \Delta)$$
   may be expressed in terms of $\hat{p} = -i\hbar d/dx$.
   Hint: you will need to employ a Taylor expansion.

   (b) T is unitary, i.e., $TT^\dagger = I$.

Turn over...

3. In a two dimensional vector space, consider the operator whose matrix, in an orthonormal basis $\{|1\rangle, |2\rangle\}$, is written

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

(a) Is σ_y Hermitian?Justify your answer

(b) Compute its eigenvalues and eigenvectors, giving their normalized expansion in terms of the $\{|1\rangle, |2\rangle\}$ basis.
Hint: To find (or guess) this basis, note that $\langle 1| \sigma_y |1\rangle = \langle 2| \sigma_y |2\rangle = 0, \langle 1| \sigma_y |2\rangle = -i$ and $\langle 2| \sigma_y |1\rangle = i$

4. The Hamiltonian operator H for a certain physical system is represented by the matrix

$$H = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

while two other observables A and B are represented by the matrices

$$A = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 2\lambda \end{pmatrix}$$

and

$$B = \begin{pmatrix} 2\mu & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \mu & 0 \end{pmatrix},$$

where λ and μ are real and nonvanishing numbers.

(a) Find the eigenvalues and eigenvectors of A, B

(b) If the system is in a state described by the state vector $\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3,$ where $c_i (i = 1, 2, 3)$ are complex constants and

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

i. Find the relationship between c_1, c_2 and c_3 so that $\mathbf{u}$ is normalized to unity

ii. Find the expectation values of $H, A,$ and B

iii. Are the $\mathbf{u}_i s$ eigenvectors of H? Why or why not?

iv. What are the possible values of the energy which can be measured if the system is described by the state vector u?