Physics of Galaxies
ANSWERS: Exercise NUMBER 7

1. *Here we will use the same equation numbering as in Chapter 3 of the notes so that we can compare them directly with the case treated there.*

Let us try wave solutions periodic in r, θ and t:

$$\sigma_1(r,\theta,t) = \sigma_{10}\exp[i(\omega t - m\theta + kr)] = \sigma_{10}\exp[i\phi] \qquad (4.32)$$

To study the wave in detail, consider the pattern generated by following the locus of constant values of $\sigma_1(r,\theta,t)$; we shall choose to follow the *peak* density $\sigma_1(r,\theta,t) = \sigma_{10}$ which is the densest part of the spiral arms – the peak of the density wave. From equation (4.32) we have at the peak $\sigma_1(r,\theta,\mathrm{t}) = \sigma_{10}$

$$\omega t - m\theta + kr = 0, \qquad (4.33)$$

Let us first trace out the pattern at a given time t by asking how the radial position r of the point of constant phase changes with the angle θ using equation (4.33),

$$\omega t - m(\theta + \Delta\theta) + k(r + \Delta r) = 0. \qquad (4.34)$$

Subtracting equation (4.33) then gives

$$\Delta r = +\frac{m}{k}\Delta\theta \qquad (4.35)$$

Remember our convention is that θ increases in the clockwise direction. Since eq. (4.35) shows that *r increases* linearly with *increasing* θ (i.e. as we go clockwise around the galaxy) we can follow an arm and see that it is spiralling *outwards* as we go around clockwise. To see that it is an *m*-armed spiral consider moving by an angle $\Delta\theta$ around the pattern at a constant value of r. To find the next point where $\sigma_1=\sigma_{10}$ the phase must change by 2π (remember that exp $(i2\pi)=1$). Using equation (4.33) gives

$$\omega t - m(\theta + \Delta\theta) + kr = 0 + 2\pi, \qquad (4.36)$$

so that, subtracting (4.33) gives

$$\Delta\theta = +\frac{2\pi}{m} \qquad (4.37)$$

This says that the pattern repeats every $2\pi/m$; thus for m=2 this repeats at θ and $\theta+2\pi/2$ $=\theta+\pi$ (of course $\theta+2(2\pi/2)=\theta+2\pi$ is exactly the same point as θ) .

The radial distance λ between adjacent arms of the spiral – the wavelength of the spiral wave – is got by keeping θ fixed and seeking the next value of r for which the phase has increased by 2π (so that again $\sigma_1=\sigma_{10}$)

$$\omega t - m\theta + k(r + \lambda) = 0 + 2\pi. \qquad (4.38)$$

Subtracting equation (4.33) from equation (4.38) and solving for the wavelength, we get

$$\lambda = \frac{2\pi}{k} \qquad (4.39)$$

so that k has its usual meaning of wavenumber.

To find the speed of rotation of the spiral pattern we differentiate equation (4.33) with respect to t at constant r,

$$\left.\frac{\partial\theta}{\partial t}\right|_r = \frac{\omega}{m} \qquad (4.40)$$

so that the spiral wave rotates clockwise at a *constant* angular speed Ω_p given by

$$\Omega_\mathrm{p} = \frac{\omega}{m}. \qquad (4.41)$$

Finally, differentiating equation (4.33) with respect to t at constant θ gives us the phase velocity of the wave travelling in the radial direction

$$\left.\frac{\partial r}{\partial t}\right|_{\theta} = -\frac{\omega}{k} \qquad (4.42)$$

so that, at a given angular position, a wave with wave-vector k advances radially into the interstellar gas with phase velocity u_{spiral} given by an expression directly comparable with the definition of phase velocity in equation (3.35)

$$u_{\text{spiral}} = -\frac{\omega}{k} = -\frac{m\Omega_{\text{p}}}{k}, \qquad (4.43)$$

We used equation (4.41) in the last step, showing that the radial velocity of the density wave is proportional to Ω_{p} as expected for the angular velocity of a spiral pattern. **[4 marks]**

The above is just a copy from the hand-out notes, Chapter 3, but with the sign on the *kr* term reversed.

The differences are:

(i) Equation (4.34) shows that the spirals move further out as we move clockwise; combining this with the unchanged result (4.40), (4.41) that the spiral pattern is still rotating clockwise (positive Ω_{p}) immediately tells us that this is a case of *leading* spiral arms. **[2 marks]**

(ii) Equation (4.43) has a negative sign, telling us that the spiral density wave now propagates inwards (rather than outwards in the –*kr* case). **[2 marks]**

The fact that the spiral density wave propagates inwards means that the formation of shocks is not likely (shocks cause compression of the ISM gas and hence star formation, giving visible spiral arms) because the speed of the spiral wave will increase as it propagates inwards (because the density increases inwards). **[2 marks]**

2. The kinetic energy K_i of the i-th atom or molecule of mass m and velocity V_i is given by

$$K_i = \frac{1}{2} m V_i^2 \ . \qquad (2.1)$$

Since they have identical mass, their *average* kinetic energy is

$$2K \approx \frac{1}{2} m \left\langle \text{v}^2 \right\rangle, \qquad (2.2)$$

where $\left\langle \text{v}^2 \right\rangle$ is their mean squared velocity. Equating the expression E_{kin} given in equation (2.2) to the average for a gas particle, 3/2*kT,* given in equation (2.1) of the question, we get

$$\frac{3}{2} kT \approx \frac{1}{2} m \left\langle \text{v}^2 \right\rangle \qquad (2.3)$$

or

$$T \approx \frac{m\left\langle \text{v}^2 \right\rangle}{3k} . \qquad \textbf{[2 marks]} \quad (2.4)$$

If we put in numbers and take $\left\langle \text{v}^2 \right\rangle^{1/2}$ to be 3,000 km s^{-1}, we get for hydrogen

$$T = \frac{1.67\times10^{-27}\times\left(3\times10^{6}\right)^{2}}{3\times1.38\times10^{-23}} \qquad \textbf{[3 marks]} \quad (2.5)$$

$$\sim 3.6\times10^{8}\ \text{K}.$$

This temperature is orders of magnitude greater than that needed to ionise the hydrogen. There would therefore be no Balmer lines to be broadened. The broadening must, therefore, be caused by *bulk* motion of the gas. **[3 marks]**

[Total marks available 18]