Week 7

1 Energy transport by convection

It is possible to calculate stellar models assuming that energy transport occurs through radiation alone. Such models, however, do not provide a realistic description of real stars because they are *unstable.*

Any theoretical model should be tested for possible instabilities, before it can be accepted as realistic. An instability normally manifests itself through the growth of small (formally infinitesimal) disturbances, with the growth through time occurring exponentially. If the characteristic growth time is less than the evolutionary time scale for the star, the disturbance may then end up significantly modifying the properties of the star. A particular type of instability is often found to operate, namely the instability corresponding to having a layer of higher density sitting on top of a layer of lower density. An extreme analogy to this instability would be a glass where a layer of mercury has been placed on top of a layer of water. This is clearly an unstable situation. Instabilities of this type are usually referred to as Rayleigh-Taylor instabilities.

In a star, this type of instability can occur if the temperature decreases too rapidly with distance from the centre. The decrease of pressure with r is determined by hydrostatic equilibrium, and is therefore largely given, and the only possibility for compensating for a rapid decrease in temperature is, according to the ideal gas law, that the density decreases slowly or even increases; this leads to the instability. From the equation of radiative transfer

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi acr^2T^3} \tag{1}$$

it follows that the temperature decreases rapidly with increasing r when the opacity is high or the luminosity is high (note that $L(r)$, the luminosity passing through radius r, corresponds to the integrated energy generated interior to the radius r).

As a result of the instability, hotter, relatively lighter fluid elements rise up through the star and cooler, relatively heavier fluid elements sink down. When the motion becomes sufficiently strong, the fluid elements are dissolved into the surrounding gas, which is then mixed. As a result, the rising elements of gas deposit their excess heat in the surroundings, and this leads to a net transport of energy out through the star. This process is known as *convection*, and the instability is called the *convective instability.* Convection is well known from everyday life, for example when air rises over a heater. Besides contributing to the energy transport, convection also leads to mixing of the parts of the star where it occurs, which has a substantial effect on the evolution of some stars.

1.1 The buoyancy force

Consider the equation of motion for an individual fluid element located at some arbitrary radius r in a star:

$$\rho(r)\frac{dv}{dt} = -\frac{dP}{dr} - \frac{Gm(r)}{r^2}\rho(r), \tag{2}$$

where the terms on the right hand side of the above equation represent the forces due to the pressure gradient and gravity, respectively. First, we consider the situation where the density of the fluid element is ρ_1, and we assume that no net force acts on the fluid element, so that

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho_1. \tag{3}$$

Now we consider the situation where the pressure gradient in the star is unchanged, but the density inside the fluid element is modified so that it has value ρ_2. It is clear from the above equation that force balance can no longer to maintained, and the force per unit volume acting on the fluid element reads

$$\rho_2\frac{dv}{dt} = -\frac{dP}{dr} - \frac{Gm(r)}{r^2}\rho_2. \tag{4}$$

Substituting for the pressure gradient term using equation (3) gives the expression

$$\rho_2 \frac{dv}{dt} = -(\rho_2 - \rho_1)\frac{Gm(r)}{r^2} = -(\rho_2 - \rho_1)g(r), \tag{5}$$

where $g(r)$ is the gravitational acceleration at radius r due to the interior mass. Hence, we see that if $\rho_2 < \rho_1$, corresponding to the fluid element having its density lowered while maintaining local pressure balance, then the fluid element will be buoyant and will experience an outwards directed acceleration.

1.2 The convective instability condition

To determine the condition for instability, we consider an element of gas (see Fig. 1) that is moved a distance Δr outwards. As indicated in Fig. 1, we denote the pressure and density in the surroundings outside the element before and after the motion by (P_1, ρ_1) and (P_2, ρ_2). The pressure and density inside the element before and after the motion are denoted by $(P_{\rm e1}, \rho_{\rm e1})$ and $(P_{\rm e2}, \rho_{\rm e2})$, respectively. When at position 1, the fluid element is assumed to have the same density and pressure as the surrounding fluid, so that $\rho_{\rm e1} = \rho_1$ and $P_{\rm e1} = P_1$. The motion of the element of gas after it has been moved a distance Δr to a point 2 is determined by the buoyancy force which, as we saw in section 1.1 above, is given by the expression

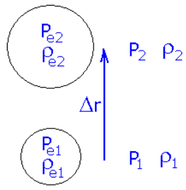

Figure 1: The motion of a convective element, from an initial position 1 to a later position 2.

$$f_{\rm buoy} = -g\,(\rho_{e2} - \rho_2) \equiv -g\Delta\rho, \tag{6}$$

where $g = -Gm(r)/r^2$ is the gravitational acceleration. Note that $f_{\rm buoy}$ is the buoyancy force per unit volume at point 2 experienced by the perturbed element of gas, and we see that it depends on the difference between the density inside the element after it has been displaced outwards and the surrounding density at its new location. If $f_{\rm buoy} > 0$, then moving the fluid element outwards by a small distance causes it to be accelerated outwards, and hence the initial displacement is amplified and the system is unstable. In the opposite case ($f_{\rm buoy} < 0$), the force is directed downwards and the element of gas returns to its original position and the system is stable (more accurately, in the stable case the element of gas actually oscillates around its original equilibrium location after being displaced).

To determine $\Delta\rho$, and hence decide whether or not the star is convectively stable or unstable, we assume that:

(i) the element of gas is always in pressure balance with the surroundings;

(ii) the motion is fast enough that there is no heat exchange between the element and its surroundings during the displacement.

From assumption (i) we have $P_{\rm e2} = P_2$. Assumption (ii) tells us that the motion takes place adiabatically. If we assume adiabatic changes then the relation between density and pressure is given by the adiabatic equation of state discussed in week 2:

$$P = K\rho^{\gamma}. \tag{7}$$

If we have an ideal gas then we can also show that

$$P = K_2 T^{\frac{\gamma}{\gamma-1}}, \tag{8}$$

where K_2 is a constant. Differentiating eqn. (7), and combining the result with eqn. (7), leads to

$$\frac{d\rho_{\rm e}}{\rho_{\rm e}} = \frac{1}{\gamma}\frac{dP_{\rm e}}{P_{\rm e}} = \frac{1}{\gamma}\frac{dP}{P} \tag{9}$$

where $d\rho_{\rm e}$ and $dP_{\rm e}$ represent the changes occurring to ρ and P inside the fluid element. From a Taylor expansion we therefore obtain (noting that $\rho_{\rm e1} = \rho_1$):

$$\begin{aligned}
\Delta\rho &= \rho_{\rm e2} - \rho_2 = \rho_{\rm e2} - \rho_{\rm e1} - (\rho_2 - \rho_1) \\
&\approx \rho_1 \frac{1}{\gamma}\frac{1}{P_1}\frac{dP}{dr}\Delta r - \frac{d\rho}{dr}\Delta r \\
&\approx \left(\frac{\rho_1}{P_1}\frac{1}{\gamma}\frac{dP}{dr} - \frac{d\rho}{dr}\right)\Delta r = \left[\left(\frac{d\rho}{dr}\right)_{\rm ad} - \frac{d\rho}{dr}\right]\Delta r, \qquad (10)
\end{aligned}$$

where we have introduced

$$\left(\frac{d\rho}{dr}\right)_{\rm ad} \equiv \frac{1}{\gamma}\frac{\rho}{P}\frac{dP}{dr}, \qquad (11)$$

the density gradient resulting from adiabatic motion in the given pressure gradient. Here it is worth noting that only changes occurring in the displaced fluid element are assumed to be adiabatic, and the background density and pressure distributions within the star do not arise because of an adiabatic equation of state.

The condition for instability is that $\Delta\rho < 0$ (see eqn. 6), i.e.

$$\frac{1}{\rho}\frac{d\rho}{dr} > \frac{1}{\gamma}\frac{1}{P}\frac{dP}{dr} \equiv \frac{d\ln\rho}{d\ln P} < \frac{1}{\gamma}. \qquad (12)$$

Note that the last inequality has changed from $>$ to $<$. This is because the first inequality in eqn. (12) compares two negative quantities, whereas the second inequality compares two positive quantities, so the inequality symbol also needs to change. This can be seen more clearly if we write the following:

$$\frac{1}{\rho}\frac{d\rho}{dr} > \frac{1}{\gamma}\frac{1}{P}\frac{dP}{dr} \equiv \frac{d\ln\rho}{dr} > \frac{1}{\gamma}\frac{d\ln P}{dr} \equiv \left|\frac{d\ln\rho}{dr}\right| < \frac{1}{\gamma}\left|\frac{d\ln P}{dr}\right| \equiv \left|\frac{d\ln\rho}{d\ln P}\right| < \frac{1}{\gamma} \qquad (13)$$

To summarise, we have considered the situation where an element of gas, sitting among an identical set of other gas elements at an initial radial location in the star labelled as '1', is perturbed upwards by a small distance Δr to a new location '2'. The initial properties inside the fluid element are $\rho_{\rm e1}$ and $P_{\rm e1}$, but it should be noted that these properties are the same as those in the surrounding fluid elements at the initial location, ρ_1 and P_1. The fluid element moves upwards without exchanging heat with its surroundings, while maintaining pressure equilibrium with those same surroundings. Hence the motion is adiabatic, and the relation between changes in the pressure and density inside the fluid element are determined by the adiabatic equation of state. The question of whether or not the fluid element wants to continue rising upwards after being moved to location '2' depend on $\Delta\rho = \rho_{\rm e2} - \rho_2$, the difference between the density inside the fluid element and its surroundings, where we require $\Delta\rho < 0$ for instability since the element needs to be less dense than its surroundings. This leads to the *instability condition* eqn. (12).

The instability condition is normally written in terms of the temperature gradient, rather than the gradient in density, because the temperature gradient is present in the basic equations of stellar structure whereas the density gradient is not. We use the ideal gas law, written in the form

$$\rho = \frac{\mu m_{\rm H} P}{k_{\rm B} T}. \qquad (14)$$

It is normally assumed that the chemical composition is independent of position (not making this assumption would lead to a criterion for convective instability called the Ledoux criterion that depends on gradients in the mean molecular weight). Assuming that the gas is ionised everywhere and chemically homogeneous, so that μ is constant, leads to the expression

$$\frac{1}{\rho}\frac{d\rho}{dr} = \frac{1}{P}\frac{dP}{dr} - \frac{1}{T}\frac{dT}{dr}. \qquad (15)$$

This leads to

$$\begin{aligned}
\left(\frac{d\rho}{dr}\right)_{\rm ad} - \frac{d\rho}{dr} &= \frac{1}{\gamma}\frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{P}\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr} \\
&= -\frac{\gamma - 1}{\gamma}\frac{\rho}{P}\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr}. \qquad (16)
\end{aligned}$$

We note that we can obtain the following expression for the adiabatic temperature gradient from equation (8)

$$\left(\frac{dT}{dr}\right)_{\rm ad} = \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{dP}{dr}. \tag{17}$$

Equation (16) can then be written

$$\left(\frac{d\rho}{dr}\right)_{\rm ad} - \frac{d\rho}{dr} = \left[\frac{dT}{dr} - \left(\frac{dT}{dr}\right)_{\rm ad}\right]\frac{\rho}{T}. \tag{18}$$

Hence, the instability condition can be written

$$\left(\frac{dT}{dr}\right)_{\rm ad} > \frac{dT}{dr}. \tag{19}$$

In other words, convective instability occurs when the adiabatic temperature gradient falls off more gently with radius than the background temperature gradient in the star. By analogy with eqn. (12), eqn. (16) can be written as (exchanging the equality symbol with the appropriate inequality for the instability):

$$\frac{d\ln T}{d\ln P} > \frac{\gamma - 1}{\gamma}. \tag{20}$$

Note that when obtaining eqn. (20) we have again reversed the inequality symbol because we have manipulated the equations in a way that changes the signs on the left-hand and right-hand sides from being negative to being positive, requiring that the inequality symbol also changes. This equation, known as the Schwartzschild condition for convective instability, shows that instability occurs if the temperature decreases too rapidly as we move outwards through the star, in perfect agreement with our simple discussion. For a fully ionised gas, $(\gamma - 1)/\gamma = 2/5$.

1.3 Where does convection occur?

To determine the circumstances under which one may expect convection, we consider a model where energy transport takes place through radiation and investigate its stability. The temperature gradient in a radiative layer is given by

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi a c r^2 T^3}. \tag{21}$$

Combining this with the equation for hydrostatic equilibrium and the ideal gas law we have

$$\frac{d\ln T}{d\ln P} = \frac{3k_{\rm B}}{16\pi a c G m_{\rm H}}\frac{\kappa}{\mu}\frac{L(r)}{m(r)}\frac{\rho}{T^3}. \tag{22}$$

From the Schwarzschild condition eqn. (20) and eqn. (22) it is evident that one may expect convection if

(i) $L(r)/m(r)$ is large. This condition corresponds to the energy generation rate per unit mass inside radius r being large. This is usually the case inside massive stars. The energy generation in such stars is a rapidly increasing function of temperature, and hence is strongly concentrated towards the centre of the star. Therefore, L/m is large and the star has a convective core.

(ii) κ is large. This is satisfied in the outer parts of relatively low mass stars on the main sequence, or more generally in stars with low surface temperatures. Here the opacity is given by Kramers law $\kappa = \kappa_0 Z(X+1)\rho T^{-3.5}$, and hence for low temperatures we have high opacity. Opacity can also increase in localised regions in radius where hydrogen is ionised, since the ionisation of hydrogen provides a means by which photons can be absorbed by matter.

(iii) ρ/T^3 is large. This is also typically satisfied in the outer regions of relatively cool stars.

(iv) $(\gamma - 1)/\gamma$ is small. i.e. The adiabatic temperature gradient is small. This is satisfied in the ionisation zone of hydrogen. i.e. Again in the outer parts of cool stars.

Thus, condition (i) predicts convection in the cores of massive stars, whereas the remaining conditions indicate a tendency for convection in the outer layers of cool stars, in particular in the envelopes of relatively cool, low mass stars on the main sequence, and in the envelopes of red giant stars. These locations of convection are summarised in Fig. 2.

1.4 Temperature gradient in convection zones

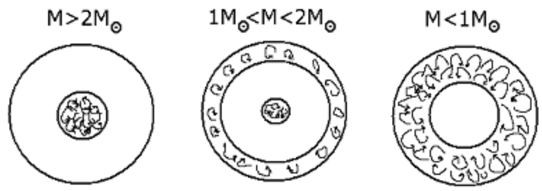

Figure 2: The occurrence of convection zones in main sequence stars. Massive stars have convective cores. Lower mass stars have convective envelopes whose depths increase as the mass decreases.

The motion of a convective element after the onset of instability is extremely difficult to describe. As a result, there is no definitive method for calculating the gas motion or the resulting rate of convective energy transport. It is generally assumed that the velocity of a convective element increases relative to the background gas to a point where new hydrodynamical instabilities set in, leading to the generation of turbulence that dissolves the element. In this way, the excess heat of the element is deposited in the surroundings, leading to energy transport. The description of such a turbulent mixing process is, and has been for a long time, the subject of on-going research. A full understanding has not been achieved so far, primarily because of shortcomings in numerical simulations of the process arising from the need to simulate the gas flows with very high spatial resolution to capture the breakdown of the convective gas flows into turbulence. We are still a long way from incorporating a complete numerical description of convection into computations of stellar models.

Fortunately, a less complete description is adequate for such computations, at least when it comes to capturing the overall properties of the star. This only requires a relation for the temperature gradient needed to transport the luminosity by convection, to replace eqn. (1) for radiative transport. It is possible to make a rough estimate of the relationship between the temperature gradient and the luminosity, and we will now derive this estimate. The result is that in most of the star the temperature gradient is only slightly steeper than the adiabatic gradient determined by eqn. (17).

We assume that a given convective element moves a distance Δr before being destroyed. As a result of the destruction, the surroundings receive the energy $\Delta u \approx \rho c_\mathrm{P} \Delta T$ per unit volume (note we are assuming that changes take place at constant pressure), where

$$\Delta T = \left[\left(\frac{dT}{dr} \right)_\mathrm{ad} - \frac{dT}{dr} \right] \Delta r \tag{23}$$

is the temperature difference between the element and the surroundings. If the mean speed of the element is v, the convective energy flux can thus be estimated as (see the discussion in lectures concerning the derivation of this expression)

$$F_\mathrm{con} \approx v \rho c_\mathrm{P} \Delta T. \tag{24}$$

To obtain an estimate of v, we equate the kinetic energy per unit volume of the element, $1/2 \rho v^2$, with the work done by the buoyancy force per unit volume over distance Δr. From eqns. (6), (10) and (16) we obtain (neglecting the factor 1/2)

$$\begin{aligned} \rho v^2 \approx f_\mathrm{buoy} \Delta r &\approx -\left[\left(\frac{d\rho}{dr} \right)_\mathrm{ad} - \frac{d\rho}{dr} \right] g \Delta r^2 \\ &\approx \frac{\rho}{T} \left[\left(\frac{dT}{dr} \right)_\mathrm{ad} - \frac{dT}{dr} \right] g \Delta r^2, . \end{aligned} \tag{25}$$

where we have used equation (18) to obtain the last line of equation (25). To simplify the notation, we introduce the dimensionless measure of the departure of the temperature gradient from its adiabatic value

$$\delta \equiv \frac{R}{T} \left[\left(\frac{dT}{dr} \right)_\mathrm{ad} - \frac{dT}{dr} \right]. \tag{26}$$

Then we finally obtain

$$F_{\rm con} \approx \rho c_{\rm P} T \delta^{3/2} \left(\frac{\Delta r}{R}\right)^2 (gR)^{1/2}, \tag{27}$$

and hence the convective luminosity (ignoring the factor of 4π)

$$L_{\rm con} \approx R^2 F_{\rm con} \approx R^3 \rho c_{\rm P} T \delta^{3/2} \left(\frac{\Delta r}{R}\right)^2 \left(\frac{g}{R}\right)^{1/2}. \tag{28}$$

In the interior of the star we can estimate $L_{\rm con}$ as

$$L_{\rm con} \approx U \delta^{3/2} \left(\frac{\Delta r}{R}\right)^2 t_{\rm dyn}^{-1}, \tag{29}$$

where we have used the relation $t_{\rm dyn} \approx (R/g)^{1/2}$, and $U \approx \rho c_{\rm P} T R^3$ is the total internal energy of the star. This equation has a simple physical interpretation. If we neglect the factor $(\Delta r/R)^2$, we have that

$$L_{\rm con} \approx (U\delta)(\delta^{1/2}/t_{\rm dyn}).$$

Here $(U\delta)$ is a measure of the internal energy that is transported; the factor δ reduces the energy transport since it is only the excess internal energy which contributes to the energy transport. Correspondingly, $(t_{\rm dyn}/\delta^{1/2})$ is the *convective time scale*, $t_{\rm con}$, which can be defined as

$$t_{\rm con} = \frac{\Delta r}{v} \approx \delta^{-1/2} \left(\frac{R}{g}\right)^{1/2} \approx \delta^{-1/2} t_{\rm dyn}, \tag{30}$$

which determines the time taken to transport the energy. $t_{\rm con}$is a dynamical time scale, but the effective gravitational acceleration is reduced, since it is only the *difference* in density which provides the force, and hence the time scale is increased by a factor $\delta^{-1/2}$.

In the case of radiative transport, the temperature gradient was determined as being sufficiently large to transport the energy by radiation. Correspondingly, in the case of convection, δ must be sufficiently large that the energy can be transported by convection. If we assume that $L = L_{\rm con}$, we obtain from eqn. (29)

$$\begin{aligned} \delta &\approx \left[\frac{L}{U}\left(\frac{\Delta r}{R}\right)^{-2} t_{\rm dyn}\right]^{2/3} \\ &\approx \left(\frac{t_{\rm dyn}}{t_{KH}}\right)^{2/3} \left(\frac{\Delta r}{R}\right)^{-4/3}, \end{aligned} \tag{31}$$

using $t_{\rm KH} \approx U/L$, where $t_{\rm KH}$ is the Kelvin-Helmholtz time scale that we discussed in the lecture of week 2. In the interior or a star, we may assume roughly that $\Delta r \approx R$. Using the values of $t_{\rm dyn}$ and $t_{\rm KH}$ for the Sun, we obtain

$$\delta \approx 5 \times 10^{-8}. \tag{32}$$

Although these estimates are uncertain, it is obvious that even an extremely small superadiabatic gradient is sufficient to transport the entire energy by convection. This simplifies the treatment of convection tremendously: at a given point in the star one determines, by means of eqn. (20), whether or not the layer is unstable. If this is the case then energy transport occurs through convection, and $\delta \approx 0$, and hence

$$\begin{aligned} \frac{dT}{dr} &= \left(\frac{dT}{dr}\right)_{\rm ad} \equiv \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{dP}{dr} \\ &= \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{Gm\rho}{r^2}. \end{aligned} \tag{33}$$

At such a point, eqn. (33) replaces the usual eqn. (1) for the temperature gradient.

From eqns. (30) and (31) we can estimate $t_{\rm con}$ as

$$\begin{aligned} t_{\rm con} &\approx \delta^{-1/2} t_{\rm dyn} \approx \left(\frac{t_{\rm KH}}{t_{\rm dyn}}\right)^{1/3} \left(\frac{\Delta r}{R}\right)^{2/3} t_{\rm dyn} \\ &= t_{\rm KH}^{1/3} t_{\rm dyn}^{2/3} \left(\frac{\Delta r}{R}\right)^{2/3}. \end{aligned} \tag{34}$$

Assuming again that $\Delta r \approx R$, we find in the case of the Sun that $t_{\rm con} \approx 1$ year. This is much shorter than the characteristic evolutionary time scale. Over a time scale not much longer than $t_{\rm con}$, matter in a convection zone must be completely mixed. Hence, we can assume that convection zones are chemically homogenous with the same chemical composition everywhere.