Week 6

1 Energy transport by radiation

Understanding in detail the interactions between matter and radiation is a subject in its own right, and an area of on-going and active research in astrophysics. This is particularly true in relatively low density environments, where photons can travel significant distances without being absorbed or scattered, and where the interactions between electromagnetic waves and ions/atoms/molecules occurs largely through line emission and absorption. These considerations are important in the interstellar medium, and in the outer atmospheric layers of stars. In the deep interiors of stars, however, we can adopt a simplified description that agrees with the complete theory when the *mean free path* of photons is very short compared to the global length scales of interest.

1.1 Mean free path and opacity

The mean free path of a photon depends on a microscopic interaction between radiation and matter. Traditionally, this interaction is described in terms of a cross section, $\sigma_{\rm R}$, such that, on average, a photon interacts with an atom if it passes within the area $\sigma_{\rm R}$ around the atom. The probability that an interaction occurs over a distance dx is given by:

$$P(dx) = n\,\sigma_{\rm R}\,dx, \tag{1}$$

and hence we can see that on average the distance travelled such that on interaction takes place is given by

$$\lambda_{\rm ph} = \frac{1}{n\sigma_{\rm R}}, \tag{2}$$

where $\lambda_{\rm ph}$ is the mean free path of a photon (i.e. the mean distance a photon travels between interactions with the particle constituents of matter). Instead of using $\lambda_{\rm ph}$ to describe the interaction between matter and radiation, it is conventional, and convenient, to use the opacity, κ, defined as

$$\kappa = \frac{1}{\rho\lambda_{\rm ph}} = \frac{n}{\rho}\sigma_{\rm R}. \tag{3}$$

Note that n/ρ is the number of atoms per unit mass; hence κ is the total cross section per unit mass. If $\sigma_{\rm R}$ and n/ρ is independent on the state of the gas (as described by its density and temperature), it follows that κ is also independent of ρ and T.

Typical values for the opacity of stellar material are of order $\kappa \approx 1$ cm^2 g^{-1} (or 0.1 m^2 kg^{-1}). With typical values of density of order $\rho \approx 1$ g cm^{-3}, the typical values of the mean free path are $\lambda \approx 1$ cm. i.e Stellar matter is very opaque, with the mean free path $\ll$ the radius of a star. Each photon is therefore subject to an enormous number of absoptions, re-emissions and scatterings as it moves from the centre of the star towards its surface. This allows us to treat the radiation transport through the star as a *diffusion process*, resulting in an enormous simplification of the problem.

1.2 The diffusion approximation

In this section we will show that the diffusive flux, $\mathbf{j}$, of transporting particles (measured in units: per unit area per unit time) between places of different particle density, n, is given by

$$\mathbf{j} = -D\nabla n \tag{4}$$

where D is the diffusion coefficient,

$$D = \frac{1}{3}v\lambda, \tag{5}$$

determined by the average values of mean velocity, v, and the mean free path, λ of the particles. This result is, of course, of much wider applicability, not just limited to the problem of radiation diffusion.

We consider the transport of particles in a time interval dt through an area dA orthogonal to the x axis at $x = 0$, as shown in Fig. 1. We work on length scales $\leq \lambda$ so that the particles move without mutually colliding. We choose the x axis to point in the direction of ∇n, so that $n = n(x)$ in the vicinity of $x = 0$, and $dn/dx > 0$. We describe the direction of motion of the particles by the angle θ that their trajectories make with the x-axis. The motion of the particles is assumed to be isotropically distributed in direction, such that when considering the motion of all particles within an infinitesimal volume, their directions of motion correspond to 4π steradians of solid angle. This also means that the particles passing through the small element of area dA are also coming from all directions, corresponding to 4π steradians of solid angle. Note that an element of solid angle is given by $d\Omega = \sin(\theta) d\theta d\phi$. When integrated over a sphere this gives $\Omega = 4\pi$ steradians.

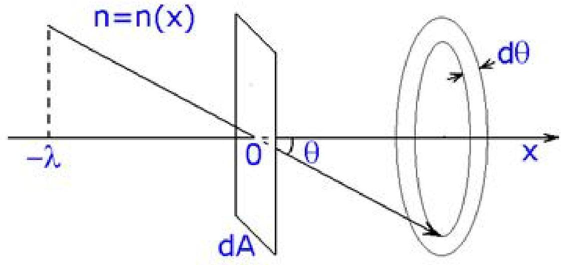

Figure 1: Geometry of photon passing through a small element of area.

Then, out of the total number of particles, the fraction of particles with directions between θ and $\theta + d\theta$ is given by

$$\frac{1}{4\pi}\int_0^{2\pi} d\phi \sin\theta d\theta = \frac{2\pi \sin(\theta) d\theta}{4\pi}.$$

On average, they come from a distance $x = \lambda \cos(\theta)$ from the small area dA through which they are passing, where the particle density is $n(x) = n(-\lambda\cos(\theta))$. Their contribution to the flux of particles through dA is given by

$$\frac{1}{2}\sin\theta d\theta \cdot n(-\lambda\cos\theta) \cdot v dt \times dA \cos\theta. \tag{6}$$

We note that in eqn. (6) the factor

$$\frac{1}{2}\sin\theta d\theta \cdot n(-\lambda\cos\theta) \cdot v dt \cdot dA$$

is the flux of particles through unit area orthogonal to the propagation direction, and the factor $\cos\theta$ accounts for the change in the projected area through which the particles are passing when approaching at an angle θ to the normal (i.e. the change in area that particles with trajectories defined by angle θ see as they move towards the small element of area dA.).

The net number of particles, dN, passing through dA from the left and right during time interval dt is obtained by integrating over all directions:

$$\begin{aligned}
dN &= \frac{1}{2} v\, dt\, dA \int_0^{\pi} n(-\lambda\cos\theta)\cdot \cos\theta \sin\theta d\theta \\
&\approx \frac{1}{2} v\, dt\, dA \int_0^{\pi} \left(n(0) + x \left.\frac{dn}{dx}\right|_{x=0} \right) \cos\theta \sin\theta d\theta \\
&= \frac{1}{2} n(0)\, v\, dt\, dA \int_0^{\pi} \cos\theta \sin\theta d\theta \\
&- \frac{1}{2} v\, \lambda \left.\frac{dn}{dx}\right|_{x=0} dt dA \int_0^{\pi} \cos^2\theta \sin\theta d\theta \\
&= -\frac{v\lambda}{3} \left.\frac{dn}{dx}\right|_{x=0} dA\, dt,
\end{aligned} \tag{7}$$

where we used a Taylor expansion of $n(x)$ about the point $x = 0$: $n(x) \approx n(0) + x\,(dn/dx)_{x=0}$. Comparing with eqn. (4), we have $|\mathbf{j}| = dN/(dA \cdot dt)$ and $\nabla n = dn/dx$, and thus the diffusion coefficient D is given by the equation (5).

1.3 Equation of radiative transport

We now apply the diffusion approximation to the energy transport by radiation. In order to obtain the diffusive flux of radiative energy, $\mathbf{F}$, we replace the number density of transporting particles,

n, by the energy density of radiation, $u_{\rm R}$, the mean velocity, v, by the speed of light, c, and λ by $\lambda_{\rm ph} = 1/(\kappa\rho)$. Owing to the spherical symmetry of the problem in a star, $\mathbf{F}$ only has a radial component, $F_r = |\mathbf{F}| = F$, and ∇n reduces to the derivative in the radial direction, $du_{\rm R}/dr$. Then eqns. (4) and (5) give immediately that

$$F = -\frac{1}{3}c\lambda_{\rm ph}\frac{du_{\rm R}}{dr} = -\frac{1}{3}\frac{c}{\kappa\rho}\frac{du_{\rm R}}{dr}. \tag{8}$$

where F is measured in units of energy per unit area per unit time (Joules per metre^2 per second). To evaluate the radiative energy flux, F, we now only need to specify the energy density of the radiation and the opacity.

The radiation energy density, $u_{\rm R}$, in the deep stellar interior is described by the black body approximation. A black body is a perfect absorber and radiator. If the radiation is in thermodynamic equilibrium with its surroundings, as for example in an enclosure or cavity whose walls are maintained at constant temperature T, each unit area of the surface emits as much radiant energy as it absorbs at each frequency, and the conditions required to obtain black body radiation are fulfilled.

The *specific intensity*, I_ν, is the amount of energy emitted per unit frequency interval per unit time flowing through unit area in unit solid angle. It may be expressed in the form

$$dE = I_\nu d\nu\, dS\, dt\, d\Omega \tag{9}$$

or equivalently

$$dE = I_\nu d\nu\, dA \cos\theta\, dt\, d\Omega. \tag{10}$$

See Fig 2 for a sketch of the geometry associated with the emission from a small element of area. For black body emission the specific intensity is given by the Planck function

$$I_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(h\nu/k_{\rm B}T\right) - 1}, \tag{11}$$

where ν is the frequency ($\nu = c/\lambda$, where λ is the wavelength), h is Planck's constant, $k_{\rm B}$ is Boltzmann's constant, and c is the speed of light. The energy emitted in unit time by unit area of a black body in all directions and across all frequencies is called the flux (or the bolometric flux) and differs from the intensity because it does not have a directional dependence. Hence to obtain the flux we integrate over all frequencies and solid angles:

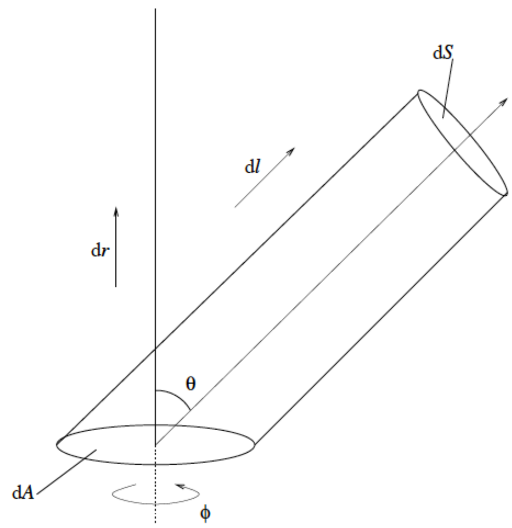

Figure 2: Geometry associated with emission from a small area, for example on the surface of a star.

$$\begin{aligned} F &= \int_{\nu=0}^{\infty}\int_{\theta=0}^{\theta=\pi/2} B_\nu \cos\theta \cdot 2\pi \sin\theta d\theta\, d\nu \\ &= \pi\int_{\nu=0}^{\nu=\infty} B_\nu d\nu \\ &= \sigma T^4, \end{aligned} \tag{12}$$

where the last equality comes from the Stefan-Boltzmann law $F = \sigma T^4$. From this expression we can deduce that the total luminosity emitted by a spherical black body of radius R is given by $L = 4\pi R^2\sigma T^4$, where

$$\sigma = \frac{2\pi^5 k_{\rm B}^4}{15c^2h^3} \tag{13}$$

is the Stefan-Boltzmann constant (the last expression can be obtained using $\int_0^\infty x^3 dx/(\exp(x) - 1) = \pi^4/15$).

We now need a relation between integrated radiation intensity

$$I = \int_0^\infty I_\nu d\nu = \int_0^\infty B_\nu d\nu = \frac{\sigma}{\pi} T^4 \tag{14}$$

and the energy density, $u_{\rm R}$ (the energy contained in the radiation field per unit volume). Using equation (9) we can write

$$dE = B_\nu d\nu\, dS\, dt\, d\Omega = B_\nu d\nu\, dS\, \frac{dl}{c}\, d\Omega \tag{15}$$

where dl represents the distance travelled at speed c by the radiation in time dt. From Fig 2 it is clear that $dV = dS\, dl$ (i.e. the product of dS and dl corresponds to a small volume element. Hence we can write

$$\frac{dE}{dV} \equiv du_{\rm R} = \frac{1}{c} B_\nu d\nu\, d\Omega. \tag{16}$$

Hence we obtain

$$u_{\rm R} = \frac{1}{c} \int_{\nu=0}^\infty \int_{\Omega=0}^{4\pi} B_\nu d\nu\, d\Omega. \tag{17}$$

Hence from equation (12) we obtain the result

$$u_{\rm R} = 4\pi \frac{I}{c} = \frac{4\sigma}{c} T^4. \tag{18}$$

Introducing

$$a = \frac{4\sigma}{c} = \frac{8\pi^5 k_{\rm B}^4}{15 c^3 h^3}, \tag{19}$$

which is known as the Stefan radiation constant, $a = 7.56 \times 10^{-16}$ J m^{-3} K^{-4}, we have

$$u_{\rm R} = aT^4, \tag{20}$$

the expression for the radiation energy density which we need for our analysis.

Returning to eqn. (8) for the diffusive flux of radiative energy, F, we get

$$F = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dr}. \tag{21}$$

If the energy transport occurs only through radiation, the total amount of energy transported through a sphere of radius r in unit time is

$$L(r) = 4\pi r^2 F.$$

We therefore arrive at the equation

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi a c r^2 T^3}, \tag{22}$$

which relates the temperature gradient to $L(r)$, and is one of the fundamental equations of stellar structure.

1.4 Opacity in stellar interiors

The interaction between radiation and matter in stars is dominated by different processes at different temperatures. At lower temperatures, where ionisation is not complete, opacity is dominated by bound-free interactions, where a photon photo-ionises an atom or an ion, leading to absorption of the photon. At somewhat higher temperatures, free-free interactions dominate. Here, an unbound electron interacts with an ion, causing it to decelerate and be deflected, and at the same time it absorbs a photon, boosting its energy. For even higher temperatures, where photo-ionisation is complete, opacity is dominated by electron scattering. Here, the electric field of a photon causes an electron to oscillate, and this oscillating electron emits a photon in a direction perpendicular to its oscillation axis and with the same frequency as the original photon. This process appears as elastic scattering between the photon and electron.

The computation of the cross section $\sigma_{\rm R}$ is in general a very complicated numerical problem. Hence, it is usual in computations of stellar models to use pre-calculated tables which tabulate how the opacity varies with ρ, T and the chemical composition. There are, however, simple approximations that can be adopted. One of these is known as *Kramers approximation*, which is given by

$$\kappa = 4.3 \times 10^{25} Z(X+1)\rho T^{-3.5} \ {\rm cm}^2{\rm g}^{-1}. \tag{23}$$

This opacity law dominates in the interiors of relatively low mass stars, where the temperature is relatively low, and bound-free and free-free interactions dominate the opacity. At higher temperatures, i.e. in the deep interiors of more massive stars, where scattering off free electrons dominates, the cross section $\sigma_{\rm R}$ for this process is independent of ρ and T; the same is true of the number, $n_{\rm e}/\rho$, of electrons per unit mass, if we assume that the gas is completely ionised. Hence the opacity is also independent of ρ and T (see eqn. 3), and is given by

$$\kappa = 0.2(X+1) \ {\rm cm}^2{\rm g}^{-1}. \tag{24}$$

1.5 The main sequence - simple scaling relations

We can estimate the luminosity of stars from the equation of radiative transport, combined with our previous estimates of the temperature and density in stars. As usual, the purpose is to get a feeling, within an order of magnitude, for the characteristic value of the luminosity, and an idea about how it varies with the global parameters of the star. Hence, in general we neglect factors of order unity.

We assume the ideal gas law; then we have an estimate for the temperature

$$T \approx \frac{GM\mu m_{\rm H}}{k_{\rm B} R} \tag{25}$$

and we estimate the mean density

$$\rho \approx \frac{M}{R^3}. \tag{26}$$

The luminosity is determined by the equation of radiative transport (eqn. 22) which we write as

$$L = -\frac{4\pi r^2 ac}{3\kappa\rho}\frac{dT^4}{dr}. \tag{27}$$

We approximate the opacity by a power law

$$\kappa = \kappa_0 \rho^p T^{-q} \tag{28}$$

(parameterising eqns. 23 & 24). Finally, we replace r by R and approximate $-dT^4/dr$ by T^4/R. Then we obtain

$$\begin{aligned} L &\approx \frac{acRT^{4+q}}{\kappa_0 \rho^{p+1}} \\ &\approx \frac{ac}{\kappa_0}\left(\frac{G\mu m_{\rm H}}{k_{\rm B}}\right)^{4+q} R^{3p-q} M^{3+q-p}. \end{aligned} \tag{29}$$

It may seem peculiar that we can calculate stellar luminosities without taking into account the processes that are responsible for energy generation. The explanation is that the star is in equilibrium. The energy production has to adjust itself to produce the amount of energy necessary to satisfy eqn. (29). This is possible because the rate of energy production is a very sensitive function of temperature, as discussed in the lecture of week 5. Hence, a small modification of the central temperature is sufficient to obtain the correct luminosity.

We can also estimate L using the energy generation rates. Comparison of the two results allows us to establish simple scaling relations which describe the location of the main sequence stars on the Hertzsprung-Russell diagram. We distinguish between two cases: lower mass stars that occupy the lower main sequence; higher mass stars that occupy the upper main sequence.

1.5.1 Lower main-sequence, relatively low mass stars

Here the temperature is relatively low, and the opacity is dominated by atomic processes, in particular bound-free transitions. Hence the opacity can be approximated by Kramers law. i.e. $p = 1$ and $q = 3.5$. Then, for stars of nearly the same chemical composition, eqn. (29) gives

$$L \propto M^{5.5} R^{-0.5} \tag{30}$$

The energy generation is dominated by the PP chain. In the lecture of week 5, when considering polytropic models, we derived the expression

$$L = A_n X^2 \mu^\alpha \frac{M^{2+\alpha}}{R^{3+\alpha}} \tag{31}$$

where A_n is a constant that depends only on the polytropic index. We note that eqn. (31) was derived using the expression

$$\epsilon_{\rm pp} = \epsilon_0 X^2 \rho T^\alpha$$

for the energy generation rate per unit mass. For stars of nearly the same polytropic index, eqn. (31) gives (using $\alpha = 4.5$)

$$L \propto M^{6.5} R^{-7.5}. \tag{32}$$

We also have a relation between stellar luminosity and effective temperature ($L = 4\pi R^2 \sigma T_{\rm eff}^4$), which gives

$$L \propto T_{\rm eff}^4 R^2. \tag{33}$$

From these relations we have

$$R \propto M^{1/7}, \quad L \propto M^{38/7}, \quad T_{\rm eff} \propto M^{9/7}. \tag{34}$$

The relation between L and $T_{\rm eff}$, represented on the H-R diagram, is thus

$$L \propto T_{\rm eff}^{38/9}. \tag{35}$$

Hence, we predict that stars of the lower main sequence shall be represented by a straight line in the $\log_{10} L$-$\log_{10} T$ plane of the H-R diagram with a slope of about 4.

1.5.2 Upper main seqeunce, relatively massive stars

Here the temperature is relatively high, and the opacity is dominated by the electron scattering. Hence, we have $p = q = 0$ in eqn. (28), and therefore

$$L \propto M^3. \tag{36}$$

The energy generation for massive stars is dominated by the CNO cycle. The energy generation rate per unit mass, $\epsilon \propto \rho T^{16}$ for stars of similar chemical composition (note we have $\epsilon_{\rm CNO} = \epsilon_1 X Z \rho T^\alpha$ with $12 < \alpha < 20$ from the lecture of week 5). We can thus use eqn. (31) but now with $\alpha = 16$, giving

$$L \propto M^{18} R^{-19}. \tag{37}$$

We can use the relation $L = 4\pi R^2 \sigma T_{\rm eff}^4$, and we have the relations

$$R \propto M^{15/19}, \quad L \propto M^3, \quad T_{\rm eff} \propto M^{27/76}. \tag{38}$$

The required relation between L and $T_{\rm eff}$ is now

$$L \propto T_{\rm eff}^{76/9}. \tag{39}$$

The predicted slope of the upper main sequence in the $\log_{10} L$-$\log_{10} T$ plot of the H-R diagram is now about 8.5, about twice as steep as for the lower main sequence.

The scaling relations developed in this section are, of course, very rough. They are based on the simplest order of magnitude estimates for the luminosity obtained in eqn. (29) using the equation for radiative energy transport eqn. (22). As we will find out in the next lecture, energy is not always transported by radiation in stars. Nevertheless, these results are not very far from agreeing with observations, and allow us to obtain an insight into the origins of the well-defined main-sequence domain of the H-R diagram, without going into detailed numerical calculations of stellar structure.