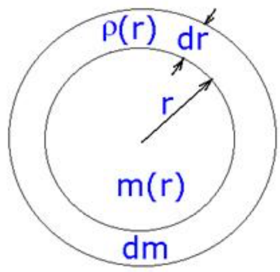
Week 3

1 Hydrostatic equilibrium

We showed in the lecture notes of week 2 that the dynamical time scale of a typical main sequence star, such as the Sun, ranges between minutes to hours. The fact that we do not observe stars undergoing structural changes on such short time scales indicates that the internal forces acting within them are in balance, with outward pressure forces balancing inward gravitational forces. A star in such a state is said to be in *hydrostatic equilibrium*. In this section we derive the equation of hydrostatic equilibrium, which is one of the four differential equations that describes the internal structure of a star.

1.1 Equation of hydrostatic support

Consider a spherical shell in a star of infinitesimal radial thickness dr located at radius r from the stellar centre. Our task is to obtain the equation of motion for this shell by determining expressions for the forces acting on it. The gravitational acceleration acting on the shell arises due to the mass of gas sitting radially interior to it, which we denote as $m(r)$. Note that Newton's sphere theorem tells us that the matter sitting exterior to the shell does not provide a net acceleration on it if the star is spherically symmetric (i.e if the density is a function of radius only). The gravitational acceleration is



$$g(r) = -\frac{Gm(r)}{r^2}. \tag{1}$$

Note that accelerations directed outwards from the stellar centre are positive, and those directed inwards are negative. The volume of the shell is $4\pi r^2 dr$, and hence the mass in the shell is

$$dm(r) = 4\pi r^2 \rho(r) dr,$$

where $\rho(r)$ is the local density. The gravitational force is

$$F_{\rm g} = -4\pi r^2 \rho(r) \frac{Gm(r)}{r^2} dr. \tag{2}$$

The outwards directed pressure force depends on the pressure difference across the width of the shell. Pressure is defined as *force per unit area*, so the pressure force can be written

$$F_{\rm P} = 4\pi r^2 P(r) - 4\pi r^2 P(r + dr). \tag{3}$$

Taking a Taylor expansion of $P(r)$, we have

$$P(r + dr) = P(r) + \left.\frac{dP}{dr}\right|_r dr + ...$$

so that eqn. (3) becomes

$$F_{\rm P} = -4\pi r^2 \frac{dP}{dr} dr. \tag{4}$$

Adding the gravitational force gives the net force acting on the shell, which we write as

$$4\pi r^2 \rho(r) dr \frac{d^2 r}{dt^2} = F_{\rm g} + F_{\rm P} \tag{5}$$

or

$$\rho \frac{d^2 r}{dt^2} = -\rho \frac{Gm}{r^2} - \frac{dP}{dr}. \tag{6}$$

If we require that the star is in equilibrium so that no net force is acting then we obtain

$$\frac{dP}{dr} = -\frac{Gm}{r^2} \rho \tag{7}$$

for the pressure gradient, which must be negative (such that pressure decreases outwards) in order to balance gravity. Equation (7) is the first equation of stellar structure, and describes the balance of pressure and gravitational forces for a star that is in hydrostatic equilibrium.

This equation must be supplemented with an equation relating $m(r)$ to the other properties of the star. This follows from the definition of $m(r)$, and the fact that the mass in the shell is $dm(r) = 4\pi\rho(r)r^2dr$, giving

$$\frac{dm}{dr} = 4\pi r^2 \rho. \tag{8}$$

This is the second equation of stellar structure.

1.2 Estimates of stellar internal pressure and temperature

From the equation of hydrostatic equilibrium (7) we can obtain an estimate for the central pressure, P_c of a star with mass M and radius radius R. We make the following approximations:
- Replace dP/dr with $-P_\mathrm{c}/R$ (note that here $dP/dr \approx (P_\mathrm{surface} - P_\mathrm{c})/R$, but $P_\mathrm{surface} = 0$).
- Replace m by M
- Replace r by R
- Replace ρ by the mean density, approximated as M/R^3.

The eqn. (7) gives

$$\frac{P_\mathrm{c}}{R} \approx \frac{GM^2}{R^5}, \tag{9}$$

or

$$P_\mathrm{c} \approx \frac{GM^2}{R^4}. \tag{10}$$

If we assume the ideal gas law

$$P = \frac{k_\mathrm{B}}{\mu m_\mathrm{H}} \rho T,$$

we may estimate the central temperature as

$$T_\mathrm{c} = \frac{\mu_\mathrm{c} m_\mathrm{H} P_\mathrm{c}}{k_\mathrm{B}\rho_\mathrm{c}} \approx \frac{G\mu_\mathrm{c} m_\mathrm{H} M}{k_\mathrm{B} R} \tag{11}$$

where μ_c is the central mean molecular weight. These estimates can be written in terms of Solar values

$$\begin{aligned} P_\mathrm{c} &\approx 1.1 \times 10^{15} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-4} \mathrm{N\,m^{-2}} \\ T_\mathrm{c} &\approx 1.9 \times 10^{7} \left(\frac{M}{M_\odot}\right) \left(\frac{R}{R_\odot}\right)^{-1} \left(\frac{\mu_\mathrm{c}}{0.85}\right) \mathrm{K} \end{aligned} \tag{12}$$

where the value of μ_c was obtained from the expression

$$\mu = \frac{4}{3 + 5X - Z}$$

derived in week 2, with $X = 0.35$ and $Z = 0.02$ (here we assume that approximately half of the original hydrogen in the Sun's core has been burned to helium, as is the case for the modern day Sun).

We should interpret these estimates as having order of magnitude accuracy only. Without any prior knowledge of the equation of hydrostatic equilibrium and the equation of state, it would be difficult to guess whether or not the central pressure of the Sun is 10^{10}, 10^{20} or 10^{30} N m^{-2}. In fact, the estimates are in reasonable agreement with the values obtained from sophisticated Solar models, which give a value for the central pressure $P_\mathrm{c} = 2.4 \times 10^{16}$ N m^{-2} and a value $T_\mathrm{c} = 1.5 \times 10^7$ K for the central temperature.

A second aspect of these estimates is that they indicate how the pressure and temperature scale with the stellar mass and radius. This dependence has a wider applicability. We shall see later several examples of how this scaling can be given a more precise meaning for particular types of simplified stellar models. And even for realistic stellar models, with detailed physics, one often finds that the scaling provided by the simple estimates are surprisingly accurate when the stellar parameters are varied. Thus, these estimates are very helpful for the interpretation of detailed numerical results.

1.3 Lower limit on the central pressure

We can obtain a *strict* lower limit on the central pressure of a star using no other assumptions other than hydrostatic equilibrium. We start with the equation of hydrostatic equilibrium

$$\begin{aligned}\frac{dP}{dr} &= -\frac{Gm}{r^2}\rho = -\frac{Gm}{4\pi r^4}4\pi r^2\rho = -\frac{Gm}{4\pi r^4}\frac{dm}{dr} \\ &= -\frac{d}{dr}\left(\frac{Gm^2}{8\pi r^4}\right) - \frac{Gm^2}{2\pi r^5}.\end{aligned} \tag{13}$$

Hence we can write

$$\frac{d}{dr}\left(P + \frac{Gm^2}{8\pi r^4}\right) = -\frac{Gm^2}{2\pi r^5} < 0. \tag{14}$$

This shows that the quantity $\Psi(r) = P + Gm^2/(8\pi r^4)$ is a decreasing function of r. At the centre, $P = P_c$. Also, eqn. (8) shows that $m \propto r^3$ for small r, so that the second term in Ψ vanishes at $r = 0$. Hence, $\Psi(0) = P_c$. At the surface we take $P = 0$. Thus, from the fact that Ψ is a decreasing function of r it follows that

$$P_c = \Psi(0) > \Psi(R) = \frac{GM^2}{8\pi R^4}, \tag{15}$$

which is the desired lower limit. This is a strict mathematical result, valid for any stellar model in hydrostatic equilibrium, regardless of its other properties, such as the equation of state, rate of energy production or transport. It also confirms that GM^2/R^4 is a characteristic value for the internal pressure of stars. It is also, however, a fairly weak limit compared to the actual Solar central pressure quoted above.

2 The virial theorem

We can derive an equation for the energetics of a star from the equation of hydrostatic equilibrium, which is important for understanding stellar evolution. Before we consider the virial theorem, however, we begin by deriving an expression for the gravitational potential energy of a spherically symmetric body, and consider the special case of uniform density sphere

2.1 Gravitational potential energy

We begin by considering the gravitational force acting on a thin spherical shell of mass dm that sits at the surface of a spherical body of mass $m(r)$ and radius r

$$F = -\frac{Gm(r)}{r^2}dm \tag{16}$$

The work done by the gravitational force due to the interior mass $m(r)$ in bringing the shell from radius r_1 to r_2 is

$$W_{r_1,r_2} = \int_{r_1}^{r_2} F\, dr = \int_{r_1}^{r_2} -\left(\frac{Gm(r)}{r^2}dm\right) dr, \tag{17}$$

which evaluates to the following (noting that the interior mass $m(r)$ is constant)

$$W_{r_1,r_2} = Gm(r)dm\left(\frac{1}{r_2} - \frac{1}{r_1}\right). \tag{18}$$

Hence, the work done in moving the shell from ∞ to the surface of the spherical body at radius r is

$$W = Gm(r)dm\left(\frac{1}{r} - \frac{1}{\infty}\right) = \frac{Gm(r)}{r}dm. \tag{19}$$

This is the amount of energy that we would need to expend in order to move the shell from radius r to ∞ against gravity, and hence the gravitational potential energy of the thin shell is

$$d\Omega = -\frac{Gm(r)}{r}dm. \tag{20}$$

To obtain the gravitational potential energy of a spherical body, we consider it as being composed of an infinite number of thin shells of mass dm and integrate over all mass shells, giving

$$\Omega = \int_M -\frac{Gm(r)}{r} dm(r). \tag{21}$$

where we write $dm(r)$ to be explicit about the radial location of the mass shell. In order to evaluate equation (27), we need to know how $m(r)$ and $dm(r)$ vary with radius.

2.1.1 Gravitational potential of uniform density sphere

Consider a uniform density sphere of mass M and radius R. The density is given by

$$\bar{\rho} = \frac{3M}{4\pi R^3}, \tag{22}$$

and hence the mass varies with radius according to

$$m(r) = \frac{4\pi}{3}\bar{\rho} r^3 \tag{23}$$

and $dm(r) = 4\pi\bar{\rho}r^2 dr$. Equation (27) can then be written as

$$\begin{aligned} \Omega &= \int_0^R -\frac{4\pi G\bar{\rho}r^2}{3} 4\pi\bar{\rho}r^2 dr \\ &= -3G\left(\frac{4\pi\bar{\rho}}{3}\right)^2 \int_0^R r^4 \\ &= -3G\left(\frac{4\pi\bar{\rho}}{3}\right)^2 \frac{R^5}{5}. \end{aligned} \tag{24}$$

Finally, substituting equation (22) into (25), we obtain

$$\Omega = -\frac{3}{5}\frac{GM^2}{R}. \tag{25}$$

The result in equation (25) demonstrates that in general we expect the gravitational potential energy to have the approximate value (correct to within a factor of order unity)

$$\Omega \approx -\frac{GM^2}{R}. \tag{26}$$

2.2 Derivation of the virial theorem

As discussed above, the gravitational potential energy of a spherical star is given by

$$\Omega = -\int_M \frac{Gm}{r} dm = -\int_0^R \frac{Gm}{r} 4\pi r^2 \rho dr = -4\pi \int_0^R Gmr\rho dr. \tag{27}$$

This may be rewritten, using the equation of hydrostatic support eqn. (7), and integrating by parts

$$\Omega = -4\pi \int_0^R \frac{Gm\rho}{r^2} r^3 dr = 4\pi \int_0^R \frac{dP}{dr} r^3 dr = \left[4\pi P r^3\right]_0^R - 3\int_0^R P 4\pi r^2 dr. \tag{28}$$

Here the integrated term vanishes, since $P = 0$ at the surface $r = R$ and $r = 0$ at the centre. Since $4\pi r^2 dr$ is a volume element, we have

$$\Omega = -3\int_V P dV, \tag{29}$$

where the integration is performed over the volume V occupied by the star. But for the ideal gas, pressure P is related to the internal energy per unit volume u as $u = 3/2P$ (equation 22 in the lecture from week 2), and we finally obtain

$$\Omega = -2U \tag{30}$$

where U is the total thermal energy of the star. This relation is called the *virial theorem*, and it applies to any spherical body that is in a state of hydrostatic equilibrium where pressure forces balance gravity. It follows that the total energy of the star is

$$E = \Omega + U = -U = \frac{1}{2}\Omega. \tag{31}$$

This shows that the total energy is negative, indicating that it is gravitationally bound and the thermal energy of the star is unable to cause it to expand to infinity.

2.3 Evolution of a star in the absence of internal energy sources

The last equation allows us to understand the evolution of stars where there are no sources of nuclear energy. Such a star in hydrostatic equilibrium will radiate some of its thermal energy through black body emission from its surface, and will contract slowly under gravity because of the loss of thermal energy. The gravitational potential energy then becomes more negative, and by eqn. (31) the same is true of the total energy of the star. Globally, however, there must be energy conservation, such that changes in the total energy of the star are balanced by losses through radiation. The rate of energy loss by the star (i.e. the luminosity) can be obtained from eqn. (31)

$$L = -\frac{dE}{dt} = -\frac{1}{2}\frac{d\Omega}{dt} \approx -\frac{1}{2}\frac{GM^2}{R^2}\frac{dR}{dt} \tag{32}$$

where the last approximation shows that any radiative losses of energy must be accompanied by contraction of the star (i.e. we must have $dR/dt < 0$). Equation (32) shows that when there is a change in the gravitational potential energy, Ω, half of this change goes into the radiation that is lost from the star. From eqn. (30) we can also see that a decrease in the gravitational potential energy results in an increase in the thermal energy, such that contraction of the star causes its temperature to rise. Differentiating eqn. (30) gives

$$\frac{dU}{dt} = -\frac{1}{2}\frac{d\Omega}{dt}. \tag{33}$$

Equations (32) and (33) show that when there is a change in the gravitational energy as the star contracts, half of this energy goes into heating the star – increasing its thermal energy, and the other half is radiated away. This demonstrates a paradoxical property of self-gravitating systems, namely that they have negative specific heat capacities: as they lose energy they get hotter.

It follows from eqn. (32) that

$$\frac{dR}{dt} \approx -2\frac{R}{t_{\rm KH}}, \tag{34}$$

where $t_{\rm KH}$ is the Kelvin-Helmoltz time scale discussed in the lecture of week 2. This demonstrates that $t_{\rm KH}$ is the characteristic time scale for the gravitational contraction of a star, and from eqn. (30) it is also the time scale for the loss of thermal energy through radiation. Hence, changes to a star that involve substantial losses or gains of energy cannot take place on time scales shorter than $t_{\rm KH}$, at least as long as hydrostatic equilibrium is approximately maintained. For changes that do occur on shorter time scales, the changes in energy must be very small and therefore must also be nearly adiabatic.

The above effects are important in understanding both the earliest phases of stellar evolution and the later stages. During the formation of a star, once the molecular cloud core has collapsed under the action of gravity, the resulting protostar undergoes slow contraction while in a state of near hydrostatic equilibrium. This is the situation as the star descends down the Hayashi track in the Hertzsprung-Russell diagram, and as it does so the centre of the star heats up until nuclear fusion reactions are initiated, at which point contraction is halted and the star joins the main sequence. In the end stages of stellar evolution, cooling of the core and slow contraction occur as nuclear fuels are exhausted. The gravitational contraction releases energy and heats up the core, until the point is reached where further nuclear reactions set in. In this case, however, the situation may be complicated by the presence elsewhere in the star of nuclear burning shells. One often also finds that the outer layers of the star expand (which requires energy to work against gravity) as the core contracts. Thus, the understanding of these evolutionary phases is less straightforward, but the virial theorem plays a central role.

When the gas cannot be regarded as ideal, or the effects of ionisation need to be taken into account, the simple equation (31) must be modified, but the principles outlined above remain applicable.