Week 2

1 Stellar time scales

Different processes occur in stars on different time scales. The dynamical time scale is the shortest one, and is the time scale on which gravitational forces would cause a star to contract in the absence of pressure forces. This is equivalent to the free fall time scale that we derived in week 1, and below we provide a simpler estimate of this time scale. We also have the thermal time scale, or the Kelvin-Helmholtz time scale, which is the time scale over which the thermal or gravitational energy content of a star is released given its observed luminosity (rate of energy release through the emission of photons). We also have the nuclear time scale, which is the time scale over which a star's nuclear fuel is consumed, again based on its current observed luminosity. These different time scales are discussed below.

1.1 The dynamical time scale

Consider a star of mass M and radius R. The gravitational acceleration at the surface is

$$g = \frac{GM}{R^2}.$$

The time required for a particle to fall a distance a under the influence of this acceleration is

$$t = \sqrt{\frac{2aR^2}{GM}}. \tag{1}$$

Taking $a = R/2$ we obtain an *estimate* of the time required for motions to occur on stellar length scales in a star's gravitational field

$$t_{\rm dyn} = \sqrt{\frac{R^3}{GM}}. \tag{2}$$

Using Solar values we may write this expression as

$$t_{\rm dyn} = 30\,{\rm min}\left(\frac{R}{R_\odot}\right)^{3/2}\left(\frac{M}{M_\odot}\right)^{-1/2}. \tag{3}$$

Hence we see that for the Sun, dynamical time scales are in the minutes to hour range. We note that stellar radii vary between $0.01\,R_\odot \le R_* \le 100\,R_\odot$, and stellar masses range between $0.1 M_\odot \le M_* \le 100 M_\odot$. Hence dynamical time scales range from seconds to years. In general, stars are not seen to undergo substantial structural changes on their dynamical time scales, indicating that their pressure and gravitational forces must be in approximate balance. In other words, stars are generally is a state of *hydrostatic equilibrium*, a point that we will come back to in week 3.

1.2 The thermal or Kelvin-Helmholtz time scale

A star that has no internal sources of energy, such as nuclear fusion reactions, can still release energy and maintain a significant luminosity. As will be demonstrated when we discuss the Virial theorem in the week 3 lecture, gravitational contraction liberates gravitational potential energy. During this process, the gravitational potential energy becomes more negative, and to ensure global conservation of energy the change in gravitational potential energy must be manifest in other forms of energy. For a star that contracts quasi-statically such that it maintains a state of near hydrostatic equilibrium, half of the liberated energy is converted into thermal energy and the other half is radiated away (we will prove this statement in the week 3 lecture). We can estimate the time scale over which the gravitational energy content of a star may be released as follows.

An estimate of the gravitational potential energy of the star is given by

$$E_{\rm grav} = -\frac{GM^2}{R}.$$

The rate at which the star loses energy is given by its luminosity, L. Hence, the time scale over which the star's gravitational energy could be released given its current luminosity is

$$t_{\rm KH} = \frac{GM^2}{LR}. \tag{4}$$

Writing this in terms of Solar values gives

$$t_{\rm KH} = 30 \text{ million years} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-1} \left(\frac{L}{L_\odot}\right)^{-1}. \tag{5}$$

Another way to think about this argument is to note that the change in gravitational energy when a star contracts from radius R to $R/2$ is $\Delta E_{\rm grav} = -GM^2/R$ (convince yourself of this statement). Hence the time for this contraction to occur is given by eqns. (4) and (5). The fact that the thermal energy content of the Sun is essentially half that of the gravitational energy means that the Kelvin-Helmholtz time scale also corresponds approximately to the time for the Sun to lose its thermal energy through the emission of radiant energy. Hence we can also equate the Kelvin-Helmholtz time scale with the thermal time scale.

In the 1800's it was not known what the Sun's energy source is. Gravitational contraction was considered to be a reasonable hypothesis, but the number obtained above for the Sun's Kelvin-Helmholtz time scale clearly limited its age to being $\approx$ 30 Myr, and by extension the Earth must also have a similar age. At the same time, geological evidence for the age of the Earth, based on estimating the time scales associated with erosion processes suggested an age of at least 300 Myr. Developments in evolutionary theory also suggested long time scales for the evolution of species that exceeded 30 Myr. The controversy was only solved in the 20$^{\rm th}$ century with the recognition that nuclear fusion provides the Sun's energy, allowing the age of the Solar System inferred from the dating of meteorites (using the fact that Uranium decays to Lead with a known half-life) to be reconciled with the Sun's luminosity. For further reading on this topic go to the following link:
`https://www.nobelprize.org/nobel_prizes/themes/physics/fusion/`.

1.3 The nuclear time scale

Stars shine throughout almost all of their lives because of the fusion of hydrogen into helium. Four hydrogen nuclei contain more mass than one helium nucleus, and the amount of energy liberated is essentially given by $\Delta E = \Delta m\, c^2$, where c is the speed of light. In the fusion of hydrogen to helium, about 0.7% of the mass is lost, and the reaction only occurs in the inner $\approx$ 10% of the mass of the star. Hence, the total amount of available energy is $\approx 7 \times 10^{-4}\, M_*$, where M_* is the stellar mass. The corresponding time scale for hydrogen burning is

$$t_{\rm nuc} = 7 \times 10^{-4}\, \frac{Mc^2}{L} \tag{6}$$

or in Solar units

$$t_{\rm nuc} = 10^{10} \text{ years} \left(\frac{M}{M_\odot}\right) \left(\frac{L}{L_\odot}\right)^{-1}. \tag{7}$$

Equation (7) gives a rough estimate of the main sequence life time of a star. Note, however, that the luminosity of a star is a strong function of its mass, so while a Solar type star lives on the main-sequence for approximately 10^{10} years, a $30\, M_\odot$ star lives for only approximately 5 million years. A $0.5\, M_\odot$ star, on the other hand, has barely had time to evolve over the current age of the Universe and will live for a much longer time than the Sun.

2 Equation of state: the ideal gas

If we want to understand the inner workings of stars, and be able to generate models that can be compared with observations, then we clearly need to understand the properties of stellar matter. These properties are described in terms of local state variables, and these can include pressure, temperature, density and internal energy, among others. (Note that quantities such as 'heat' or 'work' are not state variables, but are instead referred to as 'process functions' as they do not describe the current state of a system but rather describe how it got there.) Given the values of temperature, T, and density, ρ, for example, we can calculate all other state variables such as pressure and internal energy per unit volume. The specification of these relations constitutes the definition of the *equation of state* of the gas.

The high temperatures in stellar interiors allow for considerable simplification since the matter is normally completely ionised such that it is composed of bare nuclei and free electrons. These particles contain no internal degrees of freedom, and to a first approximation one can neglect the interactions between the particles. We refer to this gas as an *ideal gas*. An ideal gas is one in which there are no inter-particle forces, the particles occupy zero-volume, and changes in particle trajectories occur only because of elastic collisions. (Note that the assumption that electrostatic interactions between the charged particles in the plasma can be neglected arises because of the screening effect produced by the collective action of the particles in a neutral plasma. Here, the trajectories of particles are more or less ballistic, except when physical collisions occur, since the small-scale electrostatic forces are smoothed out by the screening process. This screening is called *Debye screening*.)

2.1 The ideal gas

The distribution of speeds, v, in a gas of temperature T, composed of 'classical' particles, is given by the *Maxwell distribution*:

$$f(v) = 4\pi \left(\frac{m}{2\pi k_{\rm B}T}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_{\rm B}T}\right) v^2 \tag{8}$$

where m is the mass of the particle and $k_{\rm B}$ is Boltzmann's constant. The distribution function is defined such that $f(v)dv$ gives the probability of finding a particle with speed between v and $v+dv$. Hence, $f(v)$ is normalised such that

$$\int_0^\infty f(v)dv = 1.$$

Using the Maxwell distribution, we can calculate the average value of the kinetic energy $mv^2/2$ of an individual particle:

$$\left\langle \frac{mv^2}{2} \right\rangle = \frac{\int_0^\infty \frac{mv^2}{2} f(v)dv}{\int_0^\infty f(v)dv} = \frac{3}{2}k_{\rm B}T \tag{9}$$

where $\langle X \rangle$ denotes the mean value of X (see Coursework 1 for derivation of this result).

Consider now an ideal gas of temperature T contained in a rectangular box of dimensions **a**, **b** and **c** as shown in Fig. 1. When a single particle of mass m and velocity $\mathbf{v} = (v_x, v_y, v_z)$ reflects off the wall of the box labelled with area $S = b \times c$, the particle changes its momentum by an amount $\Delta(mv_x) = 2mv_x$. A change in momentum must be accompanied by a force, since from Newton's second law we have $d\mathbf{p}/dt = \mathbf{F}$, where $\mathbf{p}$ is the momentum and $\mathbf{F}$ is the force. A small change in momentum arises because a force acts for a short period of time: $\Delta(mv) = F\Delta t$, where Δt is the time interval. The change in momentum arising when the particle hits the wall occurs because of a force F_1 applied from outside the wall:

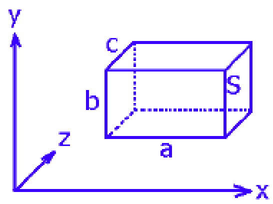

Figure 1:

$$\langle F_1 \rangle \Delta t = 2mv_x. \tag{10}$$

The time interval is the time between successive collisions with the wall, $\Delta t = 2a/v_x$. The notation $\langle\rangle$ is used to denote the fact that we are considering the time average of the force applied to the particle, and the subscript 1 reminds us that we are dealing with just a single particle. The force, F_1, is produced by the external pressure, P_1, where $\langle F_1 \rangle = P_1 S$. Combining this with our expression for Δt and eqn. (10), we obtain

$$P_1 V = mv_x^2, \tag{11}$$

where $V = a \times b \times c$, the volume of the box.

Let N be the total number of particles in volume V, and note that each of the particles contributes to the total pressure. We obtain the following expression for the total pressure

$$PV = N \left\langle m v_x^2 \right\rangle . \tag{12}$$

We also have equipartition of kinetic energy between the three degrees of freedom

$$\frac{m v_x^2}{2} = \frac{m v_y^2}{2} = \frac{m v_z^2}{2}$$

and

$$\left\langle \frac{m v_x^2}{2} \right\rangle + \left\langle \frac{m v_y^2}{2} \right\rangle + \left\langle \frac{m v_z^2}{2} \right\rangle = \left\langle \frac{m v^2}{2} \right\rangle .$$

From the Maxwell distribution eqn. (9) we have

$$\left\langle \frac{m v^2}{2} \right\rangle = \frac{3}{2} k_{\rm B} T$$

giving

$$\left\langle \frac{m v_x^2}{2} \right\rangle = \frac{1}{2} k_{\rm B} T.$$

Hence we have

$$PV = N k_{\rm B} T, \tag{13}$$

the equation of state for an ideal gas.

In the stellar case we are not concerned with a given, specific volume of gas, but instead in the microscopic properties at any location. Hence, we write eqn. (13) as

$$P = n k_{\rm B} T \tag{14}$$

where $n = N/V$ is the number of particles per unit volume. Introducing the mass density, ρ, and the dimensionless mean molecular weight, μ, we have

$$n = \frac{\rho}{\mu m_{\rm H}}$$

where $m_{\rm H}$ is the mass of the hydrogen atom ($m_{\rm H} = 1.67 \times 10^{-27}$ kg). Hence we have

$$P = \frac{k_{\rm B}}{\mu m_{\rm H}} \rho T. \tag{15}$$

The ratio $k_{\rm B}/m_{\rm H}$ is known as the *universal gas constant*

$$\mathcal{R} = \frac{k_{\rm B}}{m_{\rm H}} = 8.31 \times 10^3 \,{\rm J\,K^{-1}\,kg^{-1}}$$

so we can also the equation of state in the familiar form

$$P = \frac{\mathcal{R}}{\mu} \rho T. \tag{16}$$

2.2 Mean molecular weight

The above derivation assumed that all gas particles have the same mass, m. In practice, stellar matter consists of different elements in the form of atoms which are assumed to be fully ionised. We now generalise the equation of state to account for the different particle masses that may be present.

If the gas consists of different types of particles, each behaving like an ideal gas, with number densities, n_i, the total pressure in the gas is obtained from the sum of the partial pressures, P_i:

$$P = \sum_i P_i = \sum_i n_i k_{\rm B} T. \tag{17}$$

Note that for the same number densities, electrons make the same contribution to the pressure as nuclei despite their small masses. This is because in a collisionally relaxed gas, the equipartition of energy among the particles ensures that the electrons travel with higher speeds.

Consider a mixture of fully ionised atoms of different elements. We denote the atomic number of element i as Z_i, and its atomic weight as A_i. Its mass fraction is denoted X_i. When fully ionised, each atom contributes $Z_i + 1$ particles (Z_i electrons plus one nucleus). The number of atoms of element i per unit volume is

$$\frac{\rho X_i}{A_i m_{\rm H}}$$

and hence the total number of particles per unit volume for element i is

$$\frac{(Z_i + 1)\rho X_i}{A_i m_{\rm H}}.$$

Thus, from eqn. (17) it follows that the pressure is

$$P = \sum_i \frac{(Z_i + 1)\rho X_i}{A_i m_{\rm H}} k_{\rm B} T = \frac{k_{\rm B}}{\mu m_{\rm H}} \rho T, \tag{18}$$

where we define the mean molecular weight, μ, by

$$\mu^{-1} = \sum_i X_i \frac{Z_i + 1}{A_i}. \tag{19}$$

It is convenient to denote the mass fraction of H and He by X and Y, respectively, and the mass fraction of the remaining heavier elements by Z. This separation is useful in the stellar context because $Z \ll X$ and $Z \ll Y$ in most stellar compositions. Note that the normalisation $X + Y + Z = 1$ must hold since the combined mass fraction of all elements must sum to unity. To obtain an approximate expression for μ we take $A_1 = 1$ for hydrogen, $A_2 = 4$ for helium, and we approximate $(Z_i+1)/A_i = 1/2$ for heavy elements. Then we have

$$\mu^{-1} = 2X + \frac{3}{4}Y + \frac{1}{2}Z, \tag{20}$$

or

$$\mu = \frac{4}{8X + 3Y + 2Z} = \frac{4}{5X + 3 - Z}, \tag{21}$$

where we have used $Y = 1 - X - Z$ is the last expression.

2.3 Specific heats and adiabatic changes

The internal energy of the monotonic ideal gas is just the kinetic energy of the thermal motion of its particles. The mean internal energy per particle is $(3/2)k_{\rm B}T$, as given by eqn. (9). Thus the internal energy per unit volume is

$$u = \frac{3}{2} n k_{\rm B} T = \frac{3}{2} \frac{\rho k_{\rm B} T}{\mu m_{\rm H}} = \frac{3}{2} P. \tag{22}$$

The basic equation describing the evolution of the internal energy of the gas is the *first law of thermodynamics*. For a fixed amount of matter we can write

$$dU = dQ - PdV \tag{23}$$

where dU is the change in the internal energy of the fixed amount of matter, dQ is the amount of heat added to the matter, and V is the volume it occupies. If we now let V be the volume that corresponds to a unit of mass, then we can write $V = 1/\rho$, and $U = u/\rho$ is the internal energy per unit mass (often referred to as the specific internal). From eqn. (22) it follows that

$$U = \frac{3}{2} \frac{k_{\rm B} T}{\mu m_{\rm H}}. \tag{24}$$

We now consider a process that occurs at constant volume. We introduce the *specific heat per unit volume*, $c_{\rm V}$ as the amount of heat that has to be added to a unit mass to increase the temperature by one degree Kelvin. It follows from eqns. (23) and (24) that

$$dQ = \frac{3}{2}\frac{k_{\rm B}}{\mu m_{\rm H}}dT$$

and hence

$$c_{\rm V} = \frac{3}{2}\frac{k_{\rm B}}{\mu m_{\rm H}}. \tag{25}$$

It is also of interest to consider a process that occurs under constant pressure. To do so we consider the ideal gas law in the form $PV = Nk_{\rm B}T$, where in the case of considering changes per unit mass we have $N = 1/(\mu m_{\rm H})$ [Note: Remember that N is the number of particles. The mean mass of each particle is given by $\mu m_{\rm H}$, hence the number of particles per unit mass is $1/(\mu m_{\rm H})$.] For the changes we obtain

$$PdV + VdP = \frac{k_{\rm B}}{\mu m_{\rm H}}dT \tag{26}$$

and hence from eqn. (23)

$$\begin{aligned} dQ &= dU - VdP + \frac{k_{\rm B}}{\mu m_{\rm H}}dT \\ &= \frac{5}{2}\frac{k_{\rm B}}{\mu m_{\rm H}}dT - VdP \end{aligned} \tag{27}$$

where we have used eqn. (24). From this it follows that the specific heat at constant pressure is

$$c_{\rm P} = \frac{5}{2}\frac{k_{\rm B}}{\mu m_{\rm H}}. \tag{28}$$

A particularly important class of processes are the *adiabatic* processes, which occur without any exchange of heat - i.e. with $dQ = 0$. From eqns. (23), (24) and (25) we obtain

$$c_{\rm V}dT = -PdV. \tag{29}$$

To obtain a relation between changes in P and V (or ρ) for an adiabatic process, we use the ideal gas law in the form of eqn. (26), which can be written as (divide both sides by $PV = k_{\rm B}T/(\mu m_{\rm H})$)

$$\frac{dV}{V} + \frac{dP}{P} = \frac{dT}{T} \tag{30}$$

to write equation (29) as

$$c_{\rm V}\left(\frac{dP}{P} + \frac{dV}{V}\right) = -\frac{P}{T}dV = -\frac{k_{\rm B}}{\mu m_{\rm H}}\frac{dV}{V} = (c_{\rm V} - c_{\rm P})\frac{dV}{V} \tag{31}$$

and hence

$$\frac{dP}{P} = -\frac{c_{\rm P}}{c_{\rm V}}\frac{dV}{V} = -\gamma\frac{dV}{V} = \gamma\frac{d\rho}{\rho} \tag{32}$$

where we have introduced $\gamma = c_{\rm P}/c_{\rm V}$. For the ideal gas that we are considering, it follows from the explicit expressions for $c_{\rm V}$ and $c_{\rm P}$ that $\gamma = 5/3$. We may also write eqn. (32) as

$$\left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S = \gamma, \tag{33}$$

where the subscript indicates that the partial derivative is to be taken at constant *entropy*, i.e. without heat exchange.