

Week 10

1 Radiation pressure and the maximum masses of stars

1.1 Radiation pressure

As discussed in the lecture of week 9, the relation between the energy and momentum of particles in Einstein's theory of special relativity is given by

$$E^2 = p^2 c^2 + m_0^2 c^4, \quad (1)$$

and in the case of photons with zero rest mass we have the relations

$$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda}. \quad (2)$$

Hence, we see that photons possess momentum that can be transferred to matter during the interaction between matter and radiation in a star.

We recall that a force acting in a particular direction on a parcel of matter can be expressed as the rate of transfer of momentum to the matter, and the associated pressure is the force exerted per unit area of the surface across which the momentum transfer occurs. If the transfer of momentum occurs because of a flux of particles (number of particles crossing a surface per unit time per unit surface area of the surface) which possess momentum, then the pressure is given by the flux of particles multiplied by their momentum in the particular direction of the applied force.

In week 6, we discussed the fact that the flux of radiation passing through the surface of a black body emitter is given by (see equation (12) of the week 6 lecture notes):

$$\begin{aligned} F &= \int_{\nu=0}^{\infty} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi/2} B_{\nu} \cos \theta \cdot \sin \theta d\theta d\phi d\nu \\ &= \pi \int_{\nu=0}^{\nu=\infty} B_{\nu} d\nu \\ &= \sigma T^4. \end{aligned} \quad (3)$$

This flux is the energy per unit time passing through one side of the surface per unit area per unit time. If we now consider a small element of surface sitting inside a star at some location with temperature T , with radiation passing through it from all directions, then the radiation pressure can be written as

$$P_{\text{rad}} = \frac{1}{c} \int_{\nu=0}^{\infty} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} B_{\nu} d\nu \cos \theta^2 \sin \theta d\theta d\phi, \quad (4)$$

where the extra factor of $\cos \theta$ comes from the fact that photons passing through the surface at an angle θ to the normal transfer a correspondingly reduced momentum along the normal direction. Equation (4) can be written as

$$P_{\text{rad}} = \frac{4\pi}{3c} \int_0^{\infty} B_{\nu}(T) d\nu. \quad (5)$$

We also showed in week 6 that the energy density of black body radiation is given by (see equations 14 through to 18 in the week 6 notes)

$$u_{\text{R}} = \frac{4\pi}{c} \int_0^{\infty} B_{\nu}(T) d\nu. \quad (6)$$

Given that $u_{\text{R}} = aT^4$, as given by equation (20) in week 6, we can write

$$P_{\text{rad}} = \frac{aT^4}{3}. \quad (7)$$

1.2 Eddington luminosity

We now consider the situation where radiation pressure forces just balance gravity in a star, recognising that at high temperatures radiation pressure dominates over gas pressure given by the ideal gas law. Hence we can write

$$\frac{dP_{\text{rad}}}{dr} = \frac{4aT^3}{3} \frac{dT}{dr} = -\frac{Gm(r)}{r^2} \rho. \quad (8)$$

From week 6, we have (equation 21)

$$F = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dr}, \quad (9)$$

and

$$L = 4\pi r^2 F(r),$$

so that we can write

$$\frac{4aT^3}{3} \frac{dT}{dr} = -\frac{\kappa\rho}{c} \frac{L}{4\pi r^2}. \quad (10)$$

Hence, using eqn. (8) we find that the luminosity that allows radiation pressure to just balance gravity is given by

$$L_{\text{edd}} = \frac{4\pi cGM}{\kappa}, \quad (11)$$

where we have set $m(r) = M$, the total mass of the star. This is the Eddington luminosity, after Sir Arthur Eddington who was the first to recognise that stellar masses would be limited by radiation pressure effects.

1.3 Maximum stellar mass

In order to determine which mass of star has a luminosity equal to the Eddington luminosity given in equation (11), we need expressions for the opacity, κ , and a relationship between the stellar mass and the luminosity. At the high temperatures found in high mass stars, we expect the dominant source of opacity to be provided by electron scattering, so we have $\kappa = 0.02(1 = X) \text{ m}^2 \text{ kg}^{-1}$. For high mass stars on the zero-age main sequence, an approximate relationship between mass and luminosity is given by

$$\frac{L}{L_{\odot}} = 1.2 \times 10^5 \left(\frac{M}{30M_{\odot}} \right)^{2.4}. \quad (12)$$

Setting the luminosity in equation (12) to L_{Edd} , and substituting the right hand side into equation (11) allows us to solve for the maximum stellar mass, above which the surface layers would be removed by radiation pressure exceeding the force of gravity

$$\frac{M}{M_{\odot}} = 30^{12/7} \left(\frac{4\pi cGM_{\odot}}{1.2 \times 10^5 L_{\odot} \kappa} \right)^{5/7}. \quad (13)$$

Inserting numerical values gives $M = 135M_{\odot}$, in good agreement with the observe upper limit for stellar masses.

2 Post-main sequence evolution of high mass stars

Stars with initial masses greater than about $8 M_{\odot}$ are expected to evolve through all the stages of nuclear burning. The process begins with hydrogen burning at about $2 \times 10^7 \text{ K}$, and proceeds at progressively higher temperatures through helium, carbon, neon, oxygen and silicon burning. Silicon burning at about $3 \times 10^9 \text{ K}$ leads to a star with a central core of iron, surrounded by concentric shells containing silicon, oxygen, neon, carbon, helium and hydrogen (see figure 1). Because energy cannot be released by the thermonuclear fusion of iron (the most stable form of nuclear matter consists of nuclei near ^{56}Fe in the periodic table), the central core contracts. Initially, this contraction can be controlled by the pressure of the dense gas of degenerate electrons in the core, but as the silicon burning in the

surrounding shell deposits more iron onto the central core, the degenerate electrons in the core become increasingly relativistic. When the core mass reaches $1.4 M_{\odot}$, the electrons become ultra-relativistic and they are no longer able to support the core, which then collapses.

2.1 Core-collapse supernova explosions

Once the innermost region of the stellar core approaches $1.4 M_{\odot}$, the core is on the brink of a catastrophe, and what follows is an uncontrolled collapse of the stellar core.

To understand the onset of the collapse, we note that when a body contracts under gravity, gravitational energy is converted into internal energy and the temperature rises. If this leads to the activation of exothermic nuclear fusion, the internal energy increases, the pressure rises, and the contraction is opposed. The opposite happens if an energy-absorbing (endothermic) process is activated: energy is absorbed, the effectiveness of the pressure is diminished, and slow gravitational contraction turns into rapid gravitational collapse.

There are two energy absorbing processes which could drive the iron core of a star into an uncontrolled collapse. They are the photodissociation of atomic nuclei and the capture of electrons via inverse beta decay, which are discussed below in more detail. First, however, we consider the energetics associated with core collapse and supernova explosions.

2.1.1 Energetics

As described above, core collapse leading to a supernova occurs almost unimpeded by pressure forces because of various energy loss processes, and hence we can consider the collapse to be in free-fall. Assuming the core collapses with fixed mass of $M_c = 1.4 M_{\odot}$ from an initial radius $R_{c,i} \simeq 3000$ km to a final radius $R_{c,f} \simeq 20$ km, the gravitational energy release is simply the difference between the gravitational energies at the initial and final radii:

$$E_{\text{gr}} \simeq -\frac{GM_c^2}{R_{c,i}} + \frac{GM_c^2}{R_{c,f}} \simeq \frac{GM_c^2}{R_{c,f}} \simeq 10^{46} \text{ Joules.} \quad (14)$$

For comparison, the gravitational binding energy of the rest of the envelope is given by

$$\Omega_{\text{env}} = - \int_{M_{\text{env}}} \frac{Gm(r)dm(r)}{r} = -4\pi G \int_{R_{c,i}}^{R_{\text{env}}} m(r)\rho(r)rdr. \quad (15)$$

Taking a simple constant density model of the envelope of a Red Supergiant star of radius $R_{\text{env}} = 100 R_{\odot}$ and mass $M = 20 M_{\odot}$ surrounding a core of mass $M_c = 1.4 M_{\odot}$ and radius $R_{c,i} = 3000$ km gives a gravitational binding energy of $\Omega_{\text{env}} \simeq 10^{42}$ Joules, which is the energy required to unbind the envelope. This estimate is probably too small because in a real supergiant the mass distribution is more centrally condensed compared to our assumption of constant density, but it illustrates the fact that only a small fraction of the gravitational energy released during core collapse is required to unbind the surrounding envelope. A supernova explosion, however, does not just unbind the envelope, but also accelerates it to observed speeds in the region of $\langle v_{\text{eject}} \rangle \simeq 3 \times 10^3 \text{ km s}^{-1}$. Thus, the energy in the ejecta is given by

$$E_{\text{eject}} = \frac{1}{2} M_{\text{env}} \langle v_{\text{eject}} \rangle^2 = \frac{1}{2} \times 20 \times 2 \times 10^{30} \times (3 \times 10^6)^2 \simeq 10^{44} \text{ Joules.} \quad (16)$$

In addition, a Type II supernova has a mean bolometric luminosity $\langle L_{\text{bol}} \rangle \simeq 2 \times 10^8 L_{\odot}$ over a period of approximately three months (100 days), so the integrated energy lost in radiation is:

$$E_{\text{rad}} \simeq 2 \times 10^8 \times 3.8 \times 10^{26} \times 100 \times (60 \times 60 \times 24) \simeq 10^{42} \text{ Joules.} \quad (17)$$

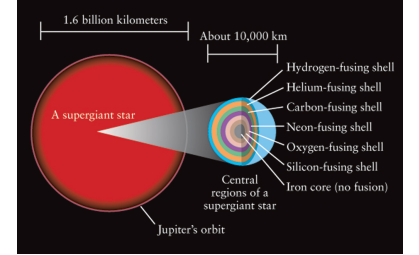


Figure 1: The Onion layer structure of the core of a red supergiant star that is close to undergoing a type II supernova explosion.

Hence, we see that approximately 1% of the gravitational energy released during the collapse of the iron core is needed to power the supernova explosion. Precisely how this 1% of gravitational energy is actually converted into the kinetic energy of the envelope – the actual mechanics of the explosion of a massive star – remains an area on on-going active research involving highly sophisticated computer simulations. Nonetheless, we provide a discussion of the main current ideas below after describing the main energy absorption processes that occur during the core collapse.

2.1.2 Nuclear photodissociation

As the iron core collapses, the temperature rises, and photons become energetic enough to initiate nuclear photodissociation. For simplicity, assume that we have the following reaction (in reality photodissociation leads to the formation of different species of nuclei)



and we see that photon energy is used to unbind atomic nuclei, leading to a net loss of thermal energy in the core that supports against gravity. If we label the particles according to their mass number, then we have

$$Q = (13m_4 + 4m_1 - m_{56})c^2 = 124.4 \text{ MeV}, \quad (19)$$

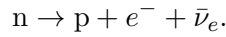
i.e. 1 kg of ${}^{56}\text{Fe}$ absorbs 2×10^{14} J of energy - equivalent to 50 kilotons of TNT. Hence, the total amount of energy that can be absorbed by this process, assuming that we have an iron core of mass $\approx 1.4 M_\odot$, is approximately

$$2 \times 10^{14} \text{ J} \times 1.4 \times 2 \times 10^{30} \approx 6 \times 10^{44} \text{ J}.$$

This is equivalent to the energy radiated by the Sun over 10^{10} years.

2.1.3 Electron capture

Neutrons are normally unstable to beta decay with a half-life of 10.25 minutes



The combined energy of the electron and anti-neutrino is 1.3 MeV (i.e. the mass-energy difference between the neutron and proton). Hence, electrons with energies up to 1.3 MeV are produced. This process of neutron beta decay can be switched off if the neutrons are immersed in a dense gas of degenerate electrons, and all of the energy states with energy up to 1.3 MeV are fully occupied. We recall from week 9 that the Fermi momentum is given by

$$p_0 = h \left(\frac{3n_e}{8\pi} \right)^{1/3}, \quad (20)$$

and the associated electron energy is

$$E_0^2 = p_0^2 c^2 + m_e^2 c^4. \quad (21)$$

If the gas has a density that corresponds to a larger Fermi momentum than this, then electrons will be present with energy > 1.3 MeV, and these can be captured by protons to form neutrons via inverse beta decay



a process called *neutronisation*. Protons in the core of evolved massive stars are bound up in nuclei, but can still capture electrons in reactions like



This occurs when the density exceeds $1.1 \times 10^{12} \text{ kg m}^{-3}$, for which the Fermi energy $E_0 = \sqrt{p^2 c^2 + m_e^2 c^4} = 3.7 \text{ MeV}$, the value needed for inverse beta decay of ${}^{56}\text{Fe}$ to occur. Normally, a ${}^{56}\text{Mn}$ nucleus decays to ${}^{56}\text{Fe}$ with a half-life of 2.6 hours, but in a stellar core it captures an electron to form ${}^{56}\text{Cr}$. Electron

capture by inverse beta decay on nuclei in a stellar core becomes very rapid when $\rho \approx 10^{14} \text{ kg m}^{-3}$. The kinetic energy of degenerate electrons is converted into the kinetic energy of electron neutrinos, ν_e , which more weakly interact with the surrounding matter and more easily escape from the core, and the loss of electrons reduces the electron degeneracy pressure. These energy absorbing processes are so effective that the collapse of the core is almost unopposed by pressure effects, and the core can collapse almost freely under gravity on a free-fall time scale (derived in week 1)

$$\tau_{\text{ff}} = \sqrt{\frac{3\pi}{32G\rho}}, \quad (24)$$

which is on the order of 1/10 of a second for $\rho \approx 10^{12} \text{ kg m}^{-3}$. The total amount of energy lost may be estimated as follows. An ^{56}Fe core with $1.4 M_{\odot}$ has $\approx 10^{57}$ electrons, which can give rise to $10^{57} \nu_e$. The average energy of the captured electrons is $\approx 10 \text{ MeV}$ when $\rho \approx 2 \times 10^{13} \text{ kg m}^{-3}$, hence the energy lost (expressed in Joules) is

$$E_{\text{cap}} \approx 10^{57} \times (10 \times 1.6 \times 10^{-13}) = 1.6 \times 10^{45} \text{ J}.$$

Hence, a key expectation is that the collapse of an iron stellar core will be accompanied by the emission of a very large flux of neutrinos. This theoretical expectation was beautifully confirmed by detection of the neutrino burst from the supernova 1987A by the Kamiokande (Japan) and Irvine-Michigan-Brookhaven (USA) neutrino experiments, as shown in figure 2.

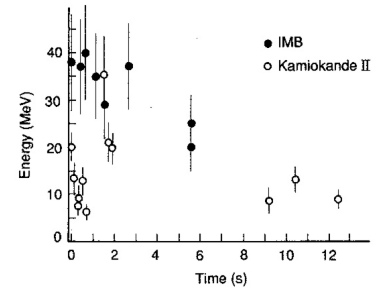


Figure 2: Neutrinos detected from supernova 1987A.

2.1.4 Explosion mechanism

The core collapse is rapid and almost unopposed until a density comparable to the density of nuclear matter is reached. The nuclear forces, which are attractive over scales $\sim 10^{-15} \text{ m}$ and repulsive on smaller scales, and the onset of neutron degeneracy, resist further compression and bring the collapse to a sudden halt. Using the fact that the radius of a nucleus is given by

$$R = r_0 A^{1/3}, \quad r_0 \approx 1.2 \times 10^{-15} \text{ m},$$

where A is the atomic mass number, we find that the density of nuclear matter

$$\rho_{\text{nuc}} \approx \frac{3Am_n}{4\pi R^3} = \frac{3m_n}{4\pi r_0^3} = 2.3 \times 10^{17} \text{ kg m}^{-3}.$$

Upon exceeding this density, the core acts as compressed spring, and rebounds strongly, setting up a shock wave that propagates outwards and travels through the material that is falling towards the centre. Early theoretical calculations suggested that this shock would be able to reverse the inwards fall of stellar material surrounding the core by transfer of kinetic energy to it and produce an outward expulsion, a *supernova*. More recent computer simulations, however, suggest that the shock may stall due to its energy being absorbed by heating the gas resulting in the photodissociation of nuclei, and indicate that the absorption of neutrinos by the overlying matter may play an important role in driving the explosion. In this *neutrino driven explosion* (see figure 3), neutrinos being emitted from the central proto-neutron star that forms at the point of the core bounce are absorbed by the layers between the neutron star and the stalled shock front via reactions such as



The absorption of the neutrinos heats the layer and increases the pressure behind the shock, driving it outwards through the surrounding envelope and leading to a supernova explosion. This process takes a few 100 milliseconds, and requires that a few percent of the neutrino energy is converted into thermal

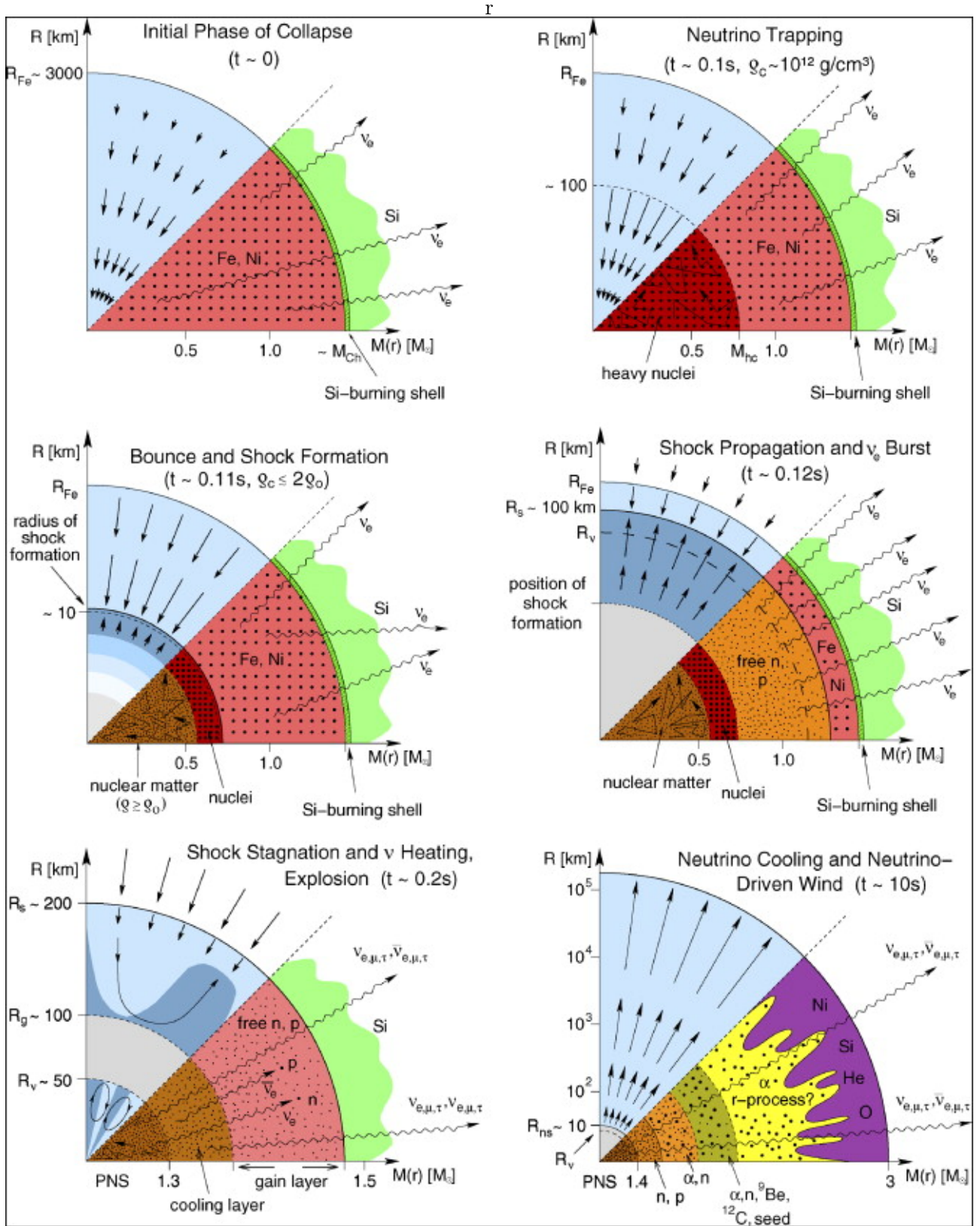


Figure 3: Schematic diagram illustrating different phases of a neutrino-driven supernova explosion, taken from Janka et al (2007), Physics Reports. Upper octant on each panel shows the dynamics at each stage, and the lower octant shows the state of matter and key processes.

energy of nucleons, leptons and photons. Among the important by-products of a supernova is the mixture of products of thermonuclear reactions accumulated around the core, and this is ejected into the interstellar medium by the explosion, and hence enriches it with heavy elements.

The collapse of the iron core of a massive star is the cause of the so-called Type II supernovae, and also of the Type Ib and Ic supernovae which differ because of the spectral emission lines present after the explosion (H and He present for Type II, He but no H lines present for Type Ib, neither He or H present in Type Ic - indicating loss of the outer envelope prior to the supernova). The collapse is expected to leave a core residue, either a *neutron star*, or an overweight neutron star that collapses to form a *black hole*. Type Ia supernovae, which are used as standard candles for measuring distances on cosmological scales, are caused by the thermonuclear detonation of a carbon-oxygen white dwarf, which increases its mass by accreting material from a close companion binary star, or by coalescing with another white dwarf star binary companion after they spiral-in due to emission of gravitational waves. If such a white dwarf exceeds the Chandrasekhar limit, then it will contract, heat up, and ignite an uncontrolled thermonuclear explosion which destroys the star completely. This arises because the effective thermostat that operates in main sequence stars to maintain a constant rate of nuclear energy generation cannot operate in a star whose support against gravity is dominated by degeneracy pressure instead of thermal pressure.

2.2 Neutron stars

A neutron star is born as a hot residue of the collapsed core of a massive star. The typical internal temperature is initially between 10^{11} and 10^{12} K. It rapidly cools by neutrino emission (the emission of neutrino-anti-neutrino pairs), and is expected to reach a temperature on the order of 10^9 K in a day and 10^8 K in 100 years. These are high temperatures according to terrestrial and solar standards, but they are low when compared to the standards set by the high densities of matter inside a neutron star. The electrons, photons and above all the neutrons, which appear to be the dominant constituents of neutron stars, are degenerate and occupy the lowest possible states consistent with Pauli's exclusion principle. The characteristic radius of a neutron star is about 15 km, which is about 2000 times smaller than the typical size of a white dwarf given by eqn. (27) in the week 9 lecture notes.

We now discuss the nature of the matter inside a neutron star, to understand why neutrons are likely to be the main constituents. Normally, the most stable nuclei are near ^{56}Fe in the periodic table. Less massive nuclei are less stable because there is a higher fraction of nuclei near the surface. More massive nuclei are less stable because the Coulomb repulsion starts to become more important. This balance changes in the presence of degenerate electrons, and as discussed above once the Fermi energy is high enough energetic electrons can cause $p \rightarrow n$ through inverse beta decay. The associated neutronisation of nuclei leads to the formation of neutron-heavy isotopes such as ^{78}Ni and ^{76}Fe that are the most stable nuclei in a degenerate electron gas when $\rho \approx 10^{14} \text{ kg m}^{-3}$. At densities $\rho > 4 \times 10^{14} \text{ kg m}^{-3}$, the phenomenon of neutron drip occurs, and neutrons start to dissociate from nuclei. The result is a dense gas in which electrons, neutrons and nuclei co-exist. The equation of state for such a mixture is fairly well understood for such a mixture for $\rho < \rho_{\text{nuc}} = 2.3 \times 10^{17} \text{ kg m}^{-3}$, but at higher densities the nuclei start to merge with another, and the state of matter is now a dense gas consisting of electrons, protons and neutrons. The equation of state now becomes very complicated to understand and calculate, since it depends not just on neutron degeneracy (so adopting an equation of state similar to that used for white dwarfs would not be accurate), but also on the complex short range interactions among the nucleons.

To understand why neutrons are the dominant constituent, let us neglect the mutual interactions between the neutrons and protons and just consider a degenerate gas of electrons, protons and neutrons. As discussed above, neutrons are prevented from decaying because the beta decay of neutrons

$$n \rightarrow p + e^- + \bar{\nu}_e$$

is blocked due to the electron degeneracy, whereas the degeneracy ensures a plentiful supply of energetic electrons that cause the inverse beta decay of protons

$$p + e^- \rightarrow n + \nu_e,$$

hence converting protons into neutrons.

2.2.1 Sizes of neutron stars

Given that we have demonstrated that neutron stars are expected to be composed primarily of neutrons, we can say that the number density of neutrons is

$$n_n \approx \frac{\rho}{m_n},$$

where m_n is the mass of the neutron (almost the same as m_H). If we make the simplifying assumption that a neutron star is supported entirely by a non-relativistic, fully-degenerate neutron gas, then we can adopt the analysis used in week 9 to obtain the mass radius relation for a white dwarf star, noting that the equation of state is the same as that for an $n = 3/2$ polytrope. For a $1.5 M_\odot$ star, this predicts the radius to be $R = 13$ km. *As an exercise, you should look through your week 9 lecture notes and verify that you can obtain the radius of a neutron star by replacing the quantities relevant for electron degeneracy pressure with those relevant for neutron degeneracy pressure.*

2.2.2 Maximum mass of a neutron star

To a first approximation, neutrons play the same supporting role in a neutron star as electrons in a white dwarf. As discussed in the lecture notes of week 9, the smaller momenta of electrons compared to protons and neutrons in a gas of a given temperature means that electron degeneracy pressure sets in at lower densities than neutron degeneracy pressure, since we require that the Fermi momentum given by

$$p_0 = \frac{h}{2} \left(\frac{3n_e}{\pi} \right)^{1/3} \quad (26)$$

is larger than that associated with the thermal motion of the particles before degeneracy pressure becomes important. Just as degenerate electrons can fail to provide enough support to prevent the collapse of a white dwarf above a certain mass, the Chandrasekhar limit, degenerate neutrons are unable to support a neutron star with a mass that exceeds a certain value.

The physics underlying the Chandrasekhar limit is clear cut. As the mass of the white dwarf approaches the limit, the central density increases and the degenerate electrons become increasingly degenerate. At the Chandrasekhar limit the electrons are ultra-relativistic and the star collapses. A similar phenomenon involving neutrons is expected in a neutron star, but there are a number of important differences. First, the interactions between the neutrons are important at the high densities found in a neutron star. Second, the gravitational fields are very strong, and Einstein's theory of gravity, not Newton's, should be used to describe the equilibrium structure of a neutron star. However, these differences do not alter the fundamental result that there is a maximum mass for a neutron star. Their main effect is to make the calculation of the maximum mass of a neutron star significantly more difficult.

We can make a simple estimate by adopting the same analysis as used in week 9 for white dwarfs, and assuming that as the mass of a neutron star increases, its neutrons become ultra-relativistic, and the equation of state changes from that of a $n = 3/2$ polytrope to that of an $n = 3$ polytrope. A self-gravitating body with such an equation of state sits on the stability/instability boundary, and has a mass that is determined entirely by the polytropic constant K . In this case, the polytropic constant is given by fundamental constants as described by eqn. (15) in the week 9 lecture notes:

$$K_2 = \frac{hc}{8} \left(\frac{3}{\pi} \right)^{1/3} \left(\frac{1+X}{2m_H} \right)^{4/3}.$$

We note that K_2 was formulated for electron degeneracy pressure, and the last factor was obtained by considering how many electrons are donated by the different elements in the star. For a gas of neutrons, this factor simply becomes unity, and the value of K_2 thus obtained can be combined with eqn. (20) from the week 4 lectures on polytropes to obtain an estimate of the maximum mass. The result of doing this gives $M = 5.26 M_\odot$. *Prove this for yourselves as an exercise!*

Note that we have neglected a large amount of complex physics in making this estimate. First, we have neglected the interactions between the neutrons. Second, and this is very important, the gravitational field in a neutron star is so strong (i.e. the escape velocity from the surface of the star is starting to approach c , the speed of light) that we need to use Einstein's theory of gravity and not Newton's to get an accurate estimate of the maximum mass of a neutron star, since Einstein's theory has the effect of making the gravitational field stronger. The result of including these effects is to reduce the maximum mass of a neutron star down to $\approx 3 M_{\odot}$. Once a neutron star exceeds this mass, it appears that it must inevitably collapse to form a black hole. Calculations suggest that the remnant from a supernova involving a progenitor star with mass $8 \leq M \leq 25 M_{\odot}$ will be a neutron star, but for higher mass progenitor stars the supernova will result in the formation of a black hole.