Week 1

1 Basic observations of stars

Astronomy is unique among the sciences as all observations, except for those conducted by a few interplanetary probes and space-craft sitting in the Solar wind, must be undertaken remotely. We cannot perform experiments on stars or undertake sample return missions from them. The main information we can obtain comes from the electromagnetic radiation we receive. If two stars orbit one another then their observed orbital motion can also provide information about their masses. In recent years, NASA's Kepler mission undertook long term photometric monitoring of more than 100,000 stars, allowing oscillation modes to be identified in some of them using asteroseismic analysis, leading to much improved knowledge about the interiors of these stars. Except for very few stars, they all appear as unresolved points of light, and hence the electromagnetic radiation we receive is integrated over their emitting surfaces. The Sun is an obvious exception, and for this particular star we can conduct very detailed observations which aid our understanding of this celestial body.

1.1 Stellar positions and distances

The mapping of stellar positions on the night sky goes back to the ancient Babylonians. When considering the relative positions of different stars on the sky, the key quantity of interest is the *angular separation*, i.e. the angles between the lines-of-sight to the stars. Traditionally these angles are measured in degrees ($^{\circ}$), or its subdivisions arcminutes ($'$) or arcseconds ($''$), defined by

$$1^{\circ} = 60' = 3600''.$$

We note that 1 radian $= (180^{\circ}/\pi) = 206265''$.

When considering stellar structure and evolution, the relative positions of the stars on the celestial sphere are of little interest, but measuring the distances to stars is of fundamental importance. For example, being able to determine observationally the intrinsic luminosity (i.e. total energy emitted per unit time) of a star relies on having an accurate measurement of its distance, and hence when comparing theoretical stellar models with observations it is important to know the distances to stars.

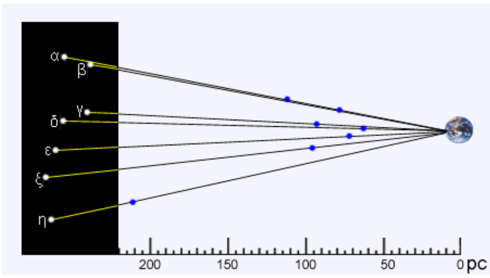

Figure 1: In 3D space the stars in Ursa Major are located far from each other. Only their projected positions on the celestial sphere give rise to an apparent pattern.

1.1.1 Parallax method

The method used to measure distances to nearby stars is called the *parallax method.* As shown by figure 2, this works by measuring the apparent position on the sky of nearby stars relative to the fixed background stars as the Earth orbits around the Sun. The change in direction to a star that occurs when the Earth moves from one side of its orbit to the other is defined to be $2 \times p$, where p is the *parallax* of the star. Hence p is the angle subtended by the radius of the Earth's orbit as viewed from the star, and hence we can write

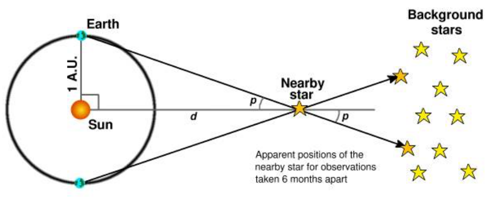

Figure 2: Diagram demonstrating how the parallax method works.

$$\tan p = \frac{1\text{AU}}{d} \quad (1)$$

where d, the distance to the star, is measured in units of AUs in equation 1. Now p is a very small angle, so

$$p \approx \frac{1\text{AU}}{d} \quad (2)$$

where p is still measured in radians. If p has a value equivalent to 1 arcsec then we know that $d = 206265$ AU (since there are 206265 arcsecs in one radian). We now define 1 parsec to be the distance to a star which has a parallax angle of 1 arcsec (i.e. 1 pc = 206253 AU). Writing eqn. (2) with p in units of arcsecs and d in parsecs gives

$$p(\text{arcsec}) = \frac{1}{d(\text{pc})} \tag{3}$$

where 1 parsec = $206253 \times 1.5 \times 10^{11}$ m = 3.09×10^{16} m.

The nearest star to the Sun has a parallax of 0.76", and hence a distance of 1.32 pc. The most accurate ground-based parallax measurements have a precision of about 0.01", allowing the distances of a few thousand stars in the Solar neighbourhood to be determined. The European Space Agency (ESA) satellite HIPPARCOS, which was launched in 1989, measured the parallaxes of more than 100,000 stars. ESA's Gaia mission, which launched in 2013, is in the process of measuring the parallaxes of more than 1.3 billion stars with a precision of 20 μ-arcsecs. A recent data release for these stars occurred in April 2018, and continued future observations will refine these measurements.

1.2 Apparent and absolute magnitudes

When measuring the brightness of a star using a telescope and a detector, the quantity that we actually measure is the *flux* (denoted by F), defined to be the amount of energy being received per unit area per unit time. This is related to the intrinsic luminosity of the star, L, by the expression

$$F = \frac{L}{4\pi d^2}$$

where d is the distance to the star.

Prior to the use of the telescope and modern detectors, astronomers adopted the *magnitude scale* to denote the brightnesses of stars, a scale that was introduced originally by the Greek astronomer Hipparchus in the 2nd century BC.

The **apparent magnitude** scale of ancient Greece was based on defining the brightest stars as having an apparent magnitude $m = +1$ and the dimmest stars (just visible to the naked eye) with a magnitude $m = +6$. Magnitude $m = +2$ stars were perceived to be half as bright as magnitude $m = +1$ stars, and magnitude $m = +3$ stars were half as bright as $m = +2$ stars. Hence the magnitude scale is logarithmic since each fixed increment along the scale corresponds to a change by a specific factor (in this case an increment of unity corresponds to a factor of 2 change in perceived brightness). A magnitude $m = +1$ star is therefore $2^5 = 32$ times brighter than a magnitude $m = +6$ star. A key point to note is that *stars with lower apparent magnitude are brighter than stars with higher apparent magnitudes!*.

This system was formalised in 1856 by defining a $m = +1$ star as being 100 times brighter than a $m = +6$ star. Therefore, stars that differ in apparent magnitude by +1 have a brightness ratio of $100^{1/5} = 2.512$. Each step on this logarithmic scale corresponds to a factor of 2.512 in brightness.

The relationship between the apparent magnitudes and fluxes received from two stars, labelled 1 and 2, is given by

$$m_2 - m_1 = 2.5\log_{10}(F_1/F_2)$$

since $\log_{10}(100) = 2$ and by definition $m_2 - m_1 = 5$, for stars that differ in flux by a factor of 100. We can rewrite the above equation as

$$m_2 - m_1 = 2.5\log_{10}(F_1) - 2.5\log_{10}(F_2),$$

and if we choose star 2 to be a calibration star whose magnitude is defined to be $m_2 = 0$, then we obtain

$$m_1 = -2.5\log_{10}(F_1) + K_1$$

where K_1 is a constant given by $K_1 = 2.5\log_{10}(F_2)$, and F_2 is the flux received from the calibration star that defines the zero point of the magnitude scale. The star Vega is used to define this zero point, so any star from which we receive the same flux on Earth as we do from Vega has zero apparent magnitude.

By definition, the **absolute magnitude** of a star is defined to be the apparent magnitude that a star would have if placed at a distance of 10 parsecs from the Earth. Absolute magnitude is denoted with an upper case M. Apparent magnitude is denoted with a lower case m. If the flux of a star measured at its current distance from Earth is denoted F_*, and the flux measured at 10 parsecs would be F_{10}, then M and m are related by

$$m - M = 2.5\log_{10}(F_{10}/F_*).$$

If we measure the actual distance to the star, d, in parsecs, then we can write

$$\frac{F_{10}}{F_*} = \frac{L}{4\pi 10^2} \times \frac{4\pi d^2}{L} = \left(\frac{d}{10}\right)^2.$$

Hence we can write

$$m - M = 5\log_{10}(d) - 5.$$

The difference between the apparent and absolute magnitudes expressed in the previous equation is often called the *distance modulus*. The Sun has an apparent magnitude $m = -27$ and an absolute magnitude $M = +4.62$.

1.3 Colour indices and surface temperature

Different stars have different colours, which depend on their temperatures; blue stars are hotter than red stars. Thus, we are not only interested in the total amount of energy emitted by a star, but also in how this energy is distributed as a function of wavelength. An indication of the distribution of energy with wavelength can be obtained by observing the star through different coloured filters, and a standard set of filters commonly used is the UBV system. It uses three filters, with sensitivity ranges:

Ultraviolet (U)	300 - 400 nm
Blue (B)	350 - 550 nm
Visual (V)	480 - 650 nm

In this system U, B and V are used to denote the apparent magnitudes measured with the corresponding filters, although $m_{\rm U}$, $m_{\rm B}$ and $m_{\rm V}$ are also used. The absolute magnitudes in these colours are denoted by $M_{\rm U}$, $M_{\rm B}$ and $M_{\rm V}$, respectively. As with the apparent magnitudes discussed above, the apparent magnitudes in U, B and V can be written as

$$U = -2.5\log_{10}(F_{\rm U}) + K_{\rm U}$$

where $F_{\rm U}$ is the flux received in the U band and $K_{\rm U}$ is a constant (similar expressions apply for B and V). To characterise the distribution of energy with wavelength, one introduces the colour indices $U - B$ and $B - V$, so that

$$U - B = 2.5\log_{10}(F_{\rm B}/F_{\rm U}) + K_{\rm U} - K_{\rm B}$$

and similarly for $B - V$. In the UBV system the constants are chosen such that $U - B = B - V = 0$ for a particular type of star (the so-called A0 dwarf stars). For the Sun, the UBV apparent magnitudes and colour indices are

U = -25.96	
B = - 26.09	U - B = 0.13
V = -26.74	B - V = 0.65

In the absence of interstellar absorption, the colour indices are independent of the distance to the star, and hence can be used to characterise its intrinsic properties.

The colour index is primarily determined by the surface temperature of a star. Hotter stars radiate more of their energy at shorter wavelengths, hence their U magnitudes tend to be low relative to their B magnitudes (remembering that magnitude decreases as luminosity increases) , and hence have a lower $U - B$ colour index than cooler stars. The same is true for the $B - V$ index. To see why this is true, consider the flux emitted from the surface of a black-body (as given by the Planck law)

$$F_\lambda = \frac{2\pi hc^2}{\lambda^5}\frac{1}{\exp\left(\frac{hc}{\lambda kT}\right) - 1}. \tag{4}$$

Note that F_λ is the energy emitted per unit area per unit time per unit wavelength interval. The quantity $F_\lambda d\lambda$ is equal to the energy emitted per unit area per unit time between wavelengths λ and $\lambda + d\lambda$. Differentiating eqn. (4) with respect to λ shows that the flux peaks at a value

$$\lambda_{\rm max} = \frac{2.8978 \times 10^6}{T}$$

such that the peak of the stellar luminosity shifts to shorter wavelengths (measured in nanometres in this expression where 1nm $= 10^{-9}$ m) at higher temperatures. By integrating eqn. (4) over wavelength one can show that the bolometric flux becomes

$$F_{\rm bol} = \sigma T^4.$$

Although stars are often reasonable approximations to black body emitters, their emission does deviate from a perfect black body. If the stellar luminosity is denoted by L, then the *effective temperature* may be defined through the expression

$$L = 4\pi R^2 \sigma T_{\rm eff}^4,$$

where R is the radius of the star. i.e. The effective temperature is defined to be the temperature that a star with bolometric luminosity L would have if it were a perfect black body.

Equation (4) can also be used to relate the colour indices to the stellar temperature. With known spectral properties of the UBV filters, eqn. (4) allows theoretical values of the $U - B$ and $B - V$ colour indices to be calculated for any assumed temperature. Matching the theoretical values with the observed values allows one to determine the surface temperature of the star, the so-called *colour temperature*. Given that stars do not radiate as perfect black bodies, in practice the actual temperature and the effective and colour temperatures will all differ from each other.

1.4 Colour-magnitude diagrams

Given measurements of the brightnesses and surface temperatures of a group of stars, as determined by their magnitudes and colour indices, one can plot these against one another and look for correlations. This was first done by E. Hertsprung and H.N. Russell, and these diagrams are commonly known as Hertsprung-Russell diagrams. Figure 3 shows such a diagram for stars that are close enough that their absolute magnitudes and luminosities could be determined. Note also that the spectral classes of the stars, OBAFGKM, corresponds closely to the effective temperature since the question of which spectral lines are visible in a star's spectrum is determined by its temperature. It is obvious that stars are not randomly distributed in the figure. The main sequence corresponds to the band running from bottom right to top left, and this is where stars spend most of their lives (cool, low luminosity stars at bottom right and hot, luminous stars at top left). The red giant stars (large, luminous and cool stars) are located at the top right. White dwarfs (hot, small, low luminosity stars) are located at bottom left. Understanding the reasons why stars are distributed in this manner on a H-R diagram is a primary goal of this lecture course.

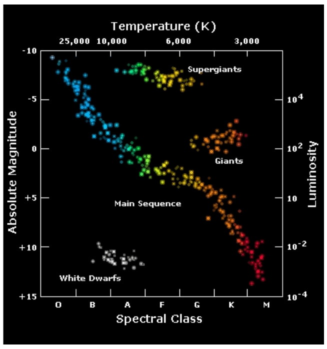

Figure 3: Hertzsrpung-Russell diagram.

Colour-magnitude diagrams are particularly useful for studying stellar clusters, since the stars are all the same distance from Earth and have the same age and initial metallicity. Figure 4 shows a colour-magnitude diagram for a cluster where the apparent visual magnitude,V, is plotted against $B - V$. The location where stars are turning off the main sequence can be used to determine the age of the cluster since cooler stars live longer than hotter stars in a predictable manner. Hence the value of the $B - V$ colour index corresponding to the turn-off tells us the age.

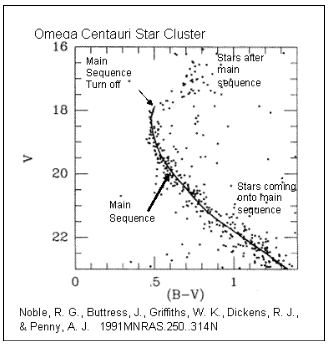

Figure 4: Colour-magnitude diagram.

2 Star formation

Stars form by the gravitational collapse of molecular cloud cores. For this to occur gravitational forces must overwhelm pressure forces and other sources of support such as magnetic fields, rotation and internal motions such as may occur if the cloud is turbulent. In the following sections we will consider the conditions required for gravitational collapse to occur and the time scale associated with the collapse.

2.1 The fluid equations

We now introduce a set of partial differential equations that describe the evolution of a non-magnetised gas under the influence of its own self-gravity and thermal pressure. These equations describe the standard laws of conservation of mass, momentum and energy. The symbols have the following meanings. ρ is the density, P is the pressure, T is the temperature, $\mathbf{v}$ is the velocity, Φ is the gravitational potential, U is the thermal energy per unit mass, F is the flux of heat due to radiation transport and ϵ is the energy generation rate (important for stars that are heated by nuclear reactions).

The continuity equation describes the conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{5}$$

The momentum equation describes the conservation of momentum

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P - \nabla \Phi. \tag{6}$$

The energy equation describes the evolution of the thermal energy per unit mass of the gas

$$\frac{\partial U}{\partial t} + (\mathbf{v} \cdot \nabla)U = -\frac{P}{\rho^2}\nabla \cdot \mathbf{v} + \epsilon - \frac{1}{\rho}\nabla \cdot \mathbf{F} \tag{7}$$

The Poisson equation describes the relation between the gravitational potential and the internal density which sources the self-gravity of the gas

$$\nabla^2 \Phi = 4\pi G \rho \tag{8}$$

Finally, the equation of state relates the pressure to the density and temperature

$$P = \frac{\mathcal{R}}{\mu}\rho T \tag{9}$$

where $\mathcal{R}$ is the gas constant and μ is the mean molecular weight. Note that we can write $\mathcal{R} = k_{\rm B}/m_{\rm H}$, where $k_{\rm B}$ is the Boltzmann constant and $m_{\rm H}$ is the mass of the hydrogen atom.

2.2 The Jeans criterion for gravitational collapse

We will now use perturbation theory to determine the conditions under which a molecular cloud core, supported against gravity by thermal pressure alone, will undergo gravitational collapse. We will use a 1-dimensional version of the equations of fluid dynamics described above to perform the analysis. The continuity equation can thus be written

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \mathbf{v})}{\partial x} = 0 \tag{10}$$

and the momentum equation can be written

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x}\right) = -\frac{\partial P}{\partial x} - \rho \frac{\partial \Phi}{\partial x}. \tag{11}$$

We will assume that the gas in the molecular cloud is isothermal, being the same temperature at all spatial locations and at all times during the evolution. This is a very good approximation for the low density conditions in the interstellar medium. Compressional heating of the gas is accompanied by efficient cooling (primarily through molecular line emission and through radiation by dust grains). Cooling during expansion is compensated for by heating due to background starlight. The isothermal approximation allows us to neglect the energy equation. The equation of state for an ideal gas is normally written as

$$P = \frac{\mathcal{R}}{\mu}\rho T. \tag{12}$$

The sound speed in a gas is defined by

$$c_{\mathrm{s}}^2 = \frac{dP}{d\rho} \tag{13}$$

and for an isothermal gas this may be written as

$$c_{\mathrm{iso}}^2 = \frac{\mathcal{R}}{\mu}T \tag{14}$$

so that the equation of state may be written in the simple form

$$P = c_{\mathrm{iso}}^2 \rho. \tag{15}$$

Finally we have the 1-dimensional version of Poisson's equation

$$\frac{\partial^2 \Phi}{\partial x^2} = 4\pi G\rho. \tag{16}$$

We note that the gravitational acceleration, $\mathbf{g}$, at any location inside the molecular cloud core is given by

$$\mathbf{g} = -\frac{\partial \Phi}{\partial x}.$$

We now consider the system to be in a state of dynamical equilibrium such that the gas is at rest (i.e.$v_x = 0$) and the pressure and gravitational forces are in balance. The density, pressure and gravitational potential in the equilibrium state are denoted by ρ_0, P_0 and Φ_0, respectively. For a general body in equilibrium these quantities will not depend on time but will depend on spatial location. The momentum equation (11) and Poisson equation (16) become

$$\begin{aligned} \frac{\partial P_0}{\partial x} &= -\rho_0 \frac{\partial \Phi_0}{\partial x} \\ \frac{\partial^2 \Phi_0}{\partial x^2} &= 4\pi G\rho_0. \end{aligned} \tag{17}$$

We now suppose that the cloud undergoes small motions about the equilibrium state, such that small perturbations develop in the velocity, pressure, density and gravitational potential

$$P = P_0 + P_1, \quad \rho = \rho_0 + \rho_1 \quad \Phi = \Phi_0 + \Phi_1, \tag{18}$$

so, for example, $P_1(x,t) \equiv P(x,t) - P_0(x)$ is the difference between the actual pressure and its equilibrium value at position x. We denote the velocity perturbation by v_x, without a subscript since the velocity is zero in the background equilibrium state. Substituting these expressions into eqns. (10), (11) and (16) gives

$$\begin{aligned} (\rho_0+\rho_1)\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x}\right) &= -\frac{\partial(P_0+P_1)}{\partial x} - (\rho_0+\rho_1)\frac{\partial(\Phi_0+\Phi_1)}{\partial x} \\ \frac{\partial(\rho_0+\rho_1)}{\partial t} &= -\frac{\partial(\rho_0+\rho_1)v_x}{\partial x} \\ \frac{\partial^2(\Phi_0+\Phi_1)}{\partial x^2} &= 4\pi G(\rho_0+\rho_1). \end{aligned} \tag{19}$$

We assume that the perturbations (the quantities with subscript '1' and the velocity v_x) are small; hence we neglect the products of two or more small quantities, since these will be even smaller. This is known as *linearising*, because we only retain equilibrium terms and terms that are linear in small quantities. This simplifies eqns. (19) to

$$\begin{aligned}\rho_0 \frac{\partial v_x}{\partial t} &= -\frac{\partial}{\partial x}(P_0 + P_1) - (\rho_0 + \rho_1)\frac{\partial \Phi_0}{\partial x} - \rho_0 \frac{\partial \Phi_1}{\partial x} \\ \frac{\partial \rho_1}{\partial t} &= -\frac{\partial}{\partial x}(\rho_0 v_x) \\ \frac{\partial^2}{\partial x^2}(\Phi_0 + \Phi_1) &= 4\pi G(\rho_0 + \rho_1). \end{aligned} \quad (20)$$

Subtracting eqns. (17) from eqns. (20) leaves a set of equations which determines the evolution of the perturbed quantities, and which are linear in those quantities

$$\begin{aligned}\rho_0 \frac{\partial v_x}{\partial t} &= -\frac{\partial P_1}{\partial x} - \rho_1 \frac{\partial \Phi_0}{\partial x} - \rho_0 \frac{\partial \Phi_1}{\partial x} \\ \frac{\partial \rho_1}{\partial t} &= -\frac{\partial}{\partial x}(\rho_0 v_x) \\ \frac{\partial^2 \Phi_1}{\partial x^2} &= 4\pi G \rho_1 \end{aligned} \quad (21)$$

We now consider the simplest possible system, which is a homogeneous cloud, infinite in all directions, so that P_0 and ρ_0 are independent of position, as too is Φ_0 by virtue of the equation of hydrostatic equilibrium given by eqn. (17). Thus from eqn. (21) we have

$$\rho_0 \frac{\partial v_x}{\partial t} = -\frac{\partial P_1}{\partial x} - \rho_0 \frac{\partial \Phi_1}{\partial x}. \quad (22)$$

Taking the divergence of this equation, and using the last of eqns. (21) to eliminate $\partial^2 \Phi_1/\partial x^2$, gives

$$\rho_0 \frac{\partial}{\partial t}\left(\frac{\partial v_x}{\partial x}\right) = -\frac{\partial^2 P_1}{\partial x^2} - 4\pi G \rho_0 \rho_1. \quad (23)$$

We assume that the gas is isothermal so that $P_1 = c_{\rm iso}^2 \rho_1$. For a uniform medium, the second expression in eqn. (21) becomes

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \frac{\partial v_x}{\partial x} \quad (24)$$

Using eqn. (24) and $P_1 = c_{\rm iso}^2 \rho_1$ to eliminate P_1 and $\partial v_x/\partial x$ from eqn. (23) we obtain

$$\frac{\partial^2 \rho_1}{\partial t^2} = c_{\rm iso}^2 \frac{\partial^2 \rho_1}{\partial x^2} + 4\pi G \rho_0 \rho_1. \quad (25)$$

This is a linear, second-oder PDE for ρ_1 with coefficients that are independent of position and time. Hence we seek plane-wave solutions of the form[1] $\rho_1(x,t) = A \exp{(ikx + i\omega t)}$, where A is a constant amplitude, k is the wavenumber of the disturbance ($k = 2\pi/\lambda$, where λ is the wavelength) and ω is the wave frequency. In this case $\partial \rho_1/\partial t = i\omega \rho_1$ and $\partial \rho_1/\partial x = ik\rho_1$. Hence, eqn. (25) can be written

$$-\omega^2 \rho_1 = -k^2 c_{\rm iso}^2 \rho_1 + 4\pi G \rho_0 \rho_1 \quad (26)$$

For a non-trivial solution (i.e. $\rho_1 \neq 0$) we obtain the dispersion relation

$$\omega^2 = k^2 c_{\rm iso}^2 - 4\pi G \rho_0. \quad (27)$$

If k and ω are real then this represents an oscillation. If the right hand side is negative, however, as it will be for sufficiently small k (equivalent to a sufficiently large wavelength for the disturbance), then ω^2 will be negative and ω will be imaginary. The values for ω may then be written

$$\omega = \pm\sqrt{k^2 c_{\rm iso}^2 - 4\pi G \rho_0} = \pm\sqrt{-1 \times (4\pi G \rho - k^2 c_{\rm iso}^2)} = \pm i \sqrt{4\pi G \rho - k^2 c_{\rm iso}^2}$$

[1] Note that here we are essentially expressing the function $\rho_1(x,t)$ as a Fourier series using complex exponentials

Hence, the cloud will be unstable because there will be a solution $\rho_1 = A \exp i\omega t \exp ikx$ which grows exponentially (and another solution which decays exponentially). Thus, the cloud is unstable to fluctuations of wavenumber k if

$$k^2 c_{\rm iso}^2 < 4\pi G\rho_0.$$

This can be rearranged to give

$$k^{-1} \equiv \frac{\lambda_{\rm J}}{2\pi} > \sqrt{\frac{c_{\rm iso}^2}{4\pi G\rho_0}}. \tag{28}$$

Now a real cloud is of finite size, so one cannot have arbitrarily large wavelengths λ. If the cloud is roughly spherical with radius R, we must have $\lambda \leq 2R$. Such a cloud is unstable to density perturbations if

$$R \geq R_{\rm J} \equiv \frac{1}{2}\sqrt{\frac{\pi c_{\rm iso}^2}{G\rho_0}}. \tag{29}$$

If we consider the cloud to be spherical then it will collapse if its mass exceeds the critical Jeans mass

$$M_{\rm J} = \frac{4}{3}\pi R_{\rm J}^3 \rho_0, \tag{30}$$

which can be written as

$$M_{\rm J} = \frac{\pi}{6}\left(\frac{\pi \mathcal{R} T_0}{\mu G}\right)^{3/2} \rho_0^{-1/2}. \tag{31}$$

Note that we can write the gas constant in terms of the Boltzmann constant, $k_{\rm B}$ and the mass of the hydrogen atom, $m_{\rm H}$

$$\mathcal{R} = \frac{k_{\rm B}}{m_{\rm H}}$$

giving

$$M_{\rm J} = \frac{\pi}{6}\left(\frac{\pi k_{\rm B} T_0}{\mu m_{\rm H} G}\right)^{3/2} \rho_0^{-1/2}. \tag{32}$$

We see that the Jeans mass is smaller for a higher density or lower temperature cloud.

2.3 A brief note on solutions to the wave equation and complex exponentials

Let us consider the equation for sound waves

$$\frac{\partial^2 \rho_1}{\partial t^2} = c_{\rm iso}^2 \frac{\partial^2 \rho_1}{\partial x^2}. \tag{33}$$

The general solution to this equation can be written as a superposition of trigonometric functions of the form

$$\rho_1(x,t) = A\cos(kx - \omega t + \phi) \tag{34}$$

where A is the amplitude, ω is the wave frequency, k is the wavenumber and ϕ is the phase. Now consider the following identity

$$\cos(X+Y) = \cos X \cos Y - \sin X \sin Y.$$

If we let $X = kx - \omega t$ and $Y = \phi$ then we have

$$\cos(kx - \omega t + \phi) = \cos(kx - \omega t)\cos\phi - \sin(kx - \omega t)\sin\phi, \tag{35}$$

from which we obtain

$$A\cos(kx - \omega t + \phi) = A\cos\phi\,\cos(kx - \omega t) - A\sin\phi\,\sin(kx - \omega t). \tag{36}$$

Hence we can write

$$A\cos(kx - \omega t + \phi) = A'\cos(kx - \omega t) + B'\sin(kx - \omega t), \tag{37}$$

where $A' = A\cos\phi$ and $B' = -A\sin\phi$. This latter form of the gneral solution may be one that you are more familiar with, and all we have done here is demonstrate its equivalence to the general form given by equation (34).

Now consider Euler's formula

$$e^{iz} = \cos z + i\sin z, \tag{38}$$

where z is a real number and $i = \sqrt{-1}$, the imaginary unit. We can write

$$Ae^{i(kx-\omega t+\phi)} = Ae^{i\phi}\, e^{i(kx-\omega t)}. \tag{39}$$

Using Euler's formula we obtain

$$Ae^{i(kx-\omega t+\phi)} = A(\cos\phi + i\sin\phi)(\cos[kx-\omega t] + i\sin[kx-\omega t]). \tag{40}$$

Expanding this expression and then collecting terms into real and imaginary parts, we obtain

$$\begin{aligned} Ae^{i(kx-\omega t+\phi)} &= (A\cos\phi\,\cos[kx-\omega t] - A\sin\phi\,\sin[kx-\omega t]) \\ &+ i(A\sin\phi\,\cos[kx-\omega t] + A\cos\phi\,\sin[kx-\omega t]). \end{aligned} \tag{41}$$

We see that

$$\begin{aligned} \mathcal{R}e\left(Ae^{i(kx-\omega t+\phi)}\right) &= A\cos\phi\,\cos[kx-\omega t] - A\sin\phi\,\sin[kx-\omega t]) \\ &= A\cos(kx-\omega t+\phi), \end{aligned} \tag{42}$$

and hence the real part of the complex exponential form is equivalent to the original general solution that we started with. This explains how the complex exponential notation used in obtaining the Jeans mass relates to the plane-wave solutions used to obtain the dispersion relation.

2.4 Free fall time

In the previous section we have determined the conditions under which a pressure supported molecular cloud core will become unstable and collapse under the influence of its own gravity. Now we will determine the typical time scale for the collapse.

We consider a spherically symmetric, homogeneous cloud with mass M and initial radius R, and assume that the cloud collapses in a state of free fall with the effects of pressure gradients being negligible. This can be justified as follows. The gravitational acceleration is $\approx (GM)/R^2$ and the acceleration due to the pressure gradient can be approximated as

$$\left|\frac{1}{\rho}\frac{\partial P}{\partial R}\right| \approx \frac{P}{\rho R} \approx \frac{kT}{\mu m_{\rm H} R}.$$

The ratio of the gravity to pressure terms is then $\propto M/(RT)$, so that for an isothermal gas the gravitational acceleration increasingly dominates over the pressure term as the collapse ensues and R decreases.

The equation of motion for a thin, spherical shell of gas located at radius r from the centre of the cloud undergoing free fall is

$$\frac{d^2r}{dt^2} = -\frac{Gm}{r^2} \tag{43}$$

where m denotes the mass contained interior to radius r. We can write the velocity of the shell

$$v(r(t)) = \frac{dr}{dt}$$

and we can rewrite the left-hand side of the equation of motion as

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}(v(r)) = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr} = \frac{1}{2}\frac{d}{dr}v^2. \tag{44}$$

The equation of motion then becomes

$$\frac{1}{2}\frac{d}{dr}v^2 = -\frac{Gm}{r^2}. \tag{45}$$

Integrating this equation for a shell of gas that sits at the cloud surface gives

$$\int \frac{1}{2}d(v^2) = -\int_R^r \frac{Gm}{r^2}dr \tag{46}$$

hence

$$v^2(r) = 2GM\left(\frac{1}{r} - \frac{1}{R}\right) \tag{47}$$

where the initial conditions are $v(R) = 0$ since the cloud starts at rest and so the velocity of the shell located at the surface of the cloud is zero. Note that we have set $m = M$, where M is the total mass of the cloud initially contained within radius R, since the mass interior to the shell does not change as the collapse ensues. Using $v = dr/dt$, eqn (47) can be written

$$\frac{dr}{dt} = \pm\left[2GM\left(\frac{1}{r} - \frac{1}{R}\right)\right]^{1/2} \tag{48}$$

Choosing the solution with $dr/dt < 0$ gives an expression from which the time taken for collapse can be determined

$$\int_0^{\tau_{\rm ff}} dt = -\int_R^0 \frac{dr}{\left[2GM\left(\frac{1}{r} - \frac{1}{R}\right)\right]^{1/2}} = -\int_R^0 \frac{dr}{\left(\frac{2GM}{R}\right)^{1/2}\left(\frac{R}{r} - 1\right)^{1/2}}. \tag{49}$$

Using the substitution

$$\zeta = \frac{r}{R}, \quad d\zeta = \frac{dr}{R}$$

we obtain

$$\int_0^{\tau_{\rm ff}} dt = -\left(\frac{2GM}{R^3}\right)^{-1/2}\int_1^0 \left(\frac{\zeta}{1-\zeta}\right)^{1/2} d\zeta. \tag{50}$$

Noting that $M/R^3 = 4\pi\rho_0/3$, where ρ_0 is the initial cloud density, this expression becomes

$$\int_0^{\tau_{\rm ff}} dt = -\left(\frac{8\pi G\rho_0}{3}\right)^{-1/2}\int_1^0 \left(\frac{\zeta}{1-\zeta}\right)^{1/2} d\zeta. \tag{51}$$

The integral on the right-hand side can be solved analytically using the substitution $\zeta = \sin^2\phi$, giving

$$\tau_{\rm ff} = \sqrt{\frac{3\pi}{32G\rho}}. \tag{52}$$