Exercise 3

Physical constants

$M_\odot = 2\times10^{30}$ kg $\quad R_\odot = 7\times10^{8}$ m $\quad M_{\rm Sun} = 4.63$ (absolute magnitude) $\quad L_\odot = 3.83\times10^{26}$ J s^{-1}
1 AU $=1.5\times10^{11}$ m $\quad G = 6.67\times10^{-11}$ N m^2 kg^{-2} $\quad \sigma = 5.7\times10^{-8}$ kg s^{-3} K^{-4} (S-B constant)
$k_{\rm B} = 1.38\times10^{-23}$ m^2 kg s^{-2} K^{-1} $\quad m_{\rm H} = 1.67\times10^{-27}$ kg

Assessed questions

Question 1

The Lane-Emden equation is given by

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

where $\rho = \rho_{\rm c}\theta^n$, $r = \alpha\xi$ and n is the polytropic index.
(i) Show that for $n = 1$, the solution to the Lane-Emden equation is given by

$$\theta = \frac{\sin\xi}{\xi}.$$

(ii) Calculate the value of ξ_1, where $\theta(\xi_1) = 0$. Calculate the value of $-(d\theta/d\xi)_{\xi=\xi_1}$.
(ii) Assuming that the polytropic model described above has a Solar mass and radius, and has mass fractions of hydrogen and helium that are 70% and 30%, respectively, determine the values of central density $\rho_{\rm c}$, pressure $P_{\rm c}$ and temperature $T_{\rm c}$, giving your answers in S.I. units.

Question 2

Energy generation in stars occurs because fusion reactions convert hydrogen to helium during their main sequence life times.
(i) Considering a head-on collision between two arbitrary nuclei with atomic numbers Z_1 and Z_2, and radii r_1 and r_2, obtain an expression for the temperature that a plasma must have in order that typical collisions between the nuclei 1 and 2 result in the nuclei physically colliding.
(ii) Using your expression, estimate the temperature required for two typical hydrogen nuclei to collide.
(iii) Assuming that atomic nuclei can be treated as constant density spheres in which the density is independent of the atomic mass, estimate the temperature required for collisions to occur between ^{12}C and ^{1}H nuclei.
(iv) Based on your answers to parts (ii) and (iii), describe in words how you expect the PP- and CNO-chains to contribute to the energy output of main sequence stars as a function of their masses.
(v) The energy generation rate per unit mass from the PP-chain is given by $\epsilon_{\rm pp} = 2.6\times10^{-37}X^2\rho T^{4.5}$ J s^{-1} kg^{-1}, and that from the CNO cycle is $\epsilon_{\rm CNO} = 7.9\times10^{-118}XZ\rho T^{16}$ J s^{-1} kg^{-1}. Above which temperature do we expect energy output from the CNO cycle to exceed that from the PP-chain per kg of stellar matter? You should assume the hydrogen mass fraction is 70% and the mass fraction of heavy elements is 2%.

Non-assessed questions

(i) Make sure that you are able to derive the Lane-Emden equation from the equations of mass conservation, hydrostatic equilibrium and the equation of state. Make sure that you can derive an expression for the Lane-Emden equation using discrete steps in ξ and θ that can be used to solve the equation numerically, and ensure that you fully understand how numerical solutions are obtained.
(ii) Make sure that you can write down the nuclear reactions that make up the PP-chains and the CNO-cycle. Make sure you understand how the energy generation rates per unit mass can be used to evaluate the total luminosity of a stellar model, and in particular how it can be used for polytropic models.