Exercise 1 Solutions

Physical constants

$\mathrm{M}_\odot = 2\times10^{30}$ kg $\mathrm{R}_\odot = 7\times10^{8}$ m $\mathrm{M}_{\mathrm{Sun}} = 4.63$ (absolute magnitude) $L_\odot = 3.83\times10^{26}$ J s^{-1}
1 AU $=1.5\times10^{11}$ m $G = 6.67\times10^{-11}$ N m^2 kg^{-2} $\sigma = 5.7\times10^{-8}$ kg s^{-3} K^{-4} (S-B constant)
$k_{\mathrm{B}} = 1.38\times10^{-23}$ m^2 kg s^{-2} K^{-1} $m_{\mathrm{H}} = 1.67\times10^{-27}$ kg

Assessed questions

Question 1

(i) An advanced civilisation living on the planet Proxima b, which orbits at a distance of 0.05 AU from its host star Proxima Centauri, is able to measure parallax angles to a precision of 0.01". What is the greatest stellar distance that they are able to measure using the parallax method? Express your answer in light years.

$$\tan p = \frac{0.05\,\mathrm{AU}}{d(\mathrm{AU})}$$

but since angle p is small we have

$$p = \frac{0.05}{d}.$$

Minimum value of p that can be measured $p = 0.01'' = 0.01/206265$ radians. Thus maximum value of d that can be measured is

$$d(\mathrm{AU}) = \frac{0.05\times206265}{0.01} = 1031324\,\mathrm{AU} = 16.3\,\mathrm{lyr}$$

12 marks

(ii) The relation between stellar mass and luminosity has been measured to be $L_* \propto M_*^{3.5}$. Proxima Centauri has a mass $M_* = 0.12\ \mathrm{M}_\odot$. Assuming that this star is able to burn all of its hydrogen to form helium, estimate its main sequence life time. How does this compare with the main sequence life time of a Solar type star?

$$L_* = L_\odot\left(\frac{M}{M_\odot}\right)^{3.5}$$

hence $L_* = (0.12)^3.5L_\odot$. From lectures we know that in burning hydrogen to helium we convert 0.7% of the rest mass into energy. Therefore the mass of the star converted to energy is

$$\Delta M = 7\times10^{-3}\times0.12\mathrm{M}_\odot.$$

Hence the total energy liberated is

$$\Delta E = \Delta M c^2 = 1.512\times10^{44}\,\mathrm{J}.$$

The nuclear time scale is equivalent to the main sequence life time and is given by

$$t_{\mathrm{nuclear}} = \frac{1.512\times10^{44}}{(0.12)^3.5L_\odot} = 2.2\times10^{13}\,\mathrm{years}.$$

This should be compared with the life time of the Sun which is $\approx 10^{10}$ years. **16 marks**
This main sequence life time estimate of Proxima is a bit too long because the luminosity obtained from the relation given above is too small compared to the real value (the scaling relation really only applies to higher mass stars), but the general principle that lower mass M-dwarf stars live for much longer than G-type stars like the Sun is nonetheless illustrated by this simple calculation.

(iii) Given that Proxima's radius has been measured to be $R_* = 0.15\ \mathrm{R}_\odot$, estimate its Kelvin-Helmholtz time scale.

Gravitational energy is given approximately by $\Omega = -GM^2/R$. Luminosity is given by $L_* = (0.12)^{3.5} L_\odot$. Hence

$$t_{\rm KH} = \frac{|\Omega|}{L_*} \approx 5 \times 10^9 {\rm yr}.$$

12 marks

Question 2
A star has an apparent magnitude 8.25 and is located at a distance 85 parsecs from the Sun.
(i) What is its absolute magnitude?
Relation between absolute and apparent magnitude

$$m - M = \log_{10}(F_{10}/F_*) = 5\log_{10}(d) - 5$$

where F_{10} is flux received from 10 pc and F_* is flux received from current position. This gives $M = 3.6$.
12 marks

(ii) What is its luminosity?
Here we use the relation between absolute magnitudes for the Sun and the star

$$M_* - M_{\rm sun} = 2.5\log_{10}(F_{\rm sun}/F_*) = 2.5\log_{10}(L_{\rm sun}/L_*).$$

Using $M_* = 3.6$ and $M_{\rm sun} = 4.63$ gives $L_* = 2.58 L_{\rm sun} = 9.88 \times 10^{26}$ J s^{-1}. **12 marks**

(iii) The star's radius is 0.5 $R_\odot$. What is its effective temperature?
Here we use

$$L_* = 4\pi R_*^2 \sigma T_*^4$$

Dividing by the same expression for the Sun a rearranging gives

$$T_*^4 = T_{\rm sun}^4 \frac{L_*}{L_{\rm sun}} \left(\frac{R_*}{R_{\rm sun}}\right)^{-2},$$

from which we obtain $T_* \approx 10354$ K. **12 marks**
Obviously this is a bit hot for a star that is smaller than the Sun!

Question 3
Consider a molecular cloud core composed entirely of molecular hydrogen (mean molecular weight $\mu = 2$) which has a uniform density and a temperature $T = 10$ K. The number density of molecules $n = 10^{12}$ m^{-3}, and the radius of the cloud $R = 3 \times 10^{16}$ m.
(i) Determine whether or not the cloud will be unstable to gravitational collapse.
The Jeans mass is given by

$$M_{\rm J} = \left(\frac{\pi k_{\rm B} T}{4\mu m_{\rm H} G\rho}\right)^{3/2} \rho,$$

which gives a value of $M_{\rm J} = 1.86 \times 10^{29}$ kg. The Solar mass is $M_\odot = 2 \times 10^{30}$ kg, so we see that the Jeans mass is about one quarter of that. The mass in the cloud is given by $M_{\rm cloud} = (4\pi/3)R^3\rho = 3.8 \times 10^{35}$ kg which is approximately 2×10^5 $M_\odot$. Clearly the cloud will collapse since $M_{\rm cloud} > M_{\rm J}$. **12 marks**

(ii) Calculate the free fall time of the cloud.
Free fall time is given by

$$\tau_{\rm ff} = \sqrt{\frac{3\pi}{32 G\rho}}$$

which gives $\tau_{\rm ff} \approx 38,000$ yr. **12 marks**

Non-assessed questions
(i) Make sure that you are able to follow the derivation of the Jeans mass and the free-fall time in

detail, as given in the online lecture notes.
(ii) In the lecture of week 2 we stated that the distribution of particle velocities in an ideal gas follows the Maxwell distribution

$$f(v) = 4\pi \left(\frac{m}{2\pi k_{\rm B} T} \right)^{3/2} \exp\left(-\frac{mv^2}{2k_{\rm B}T} \right) v^2 \tag{1}$$

where m is the mass of the particle and $k_{\rm B}$ is Boltzmann's constant. Prove that

$$\int_0^\infty f(v) dv = 1,$$

and prove that

$$\left\langle \frac{mv^2}{2} \right\rangle = \frac{\int_0^\infty \frac{mv^2}{2} f(v) dv}{\int_0^\infty f(v) dv} = \frac{3}{2} k_{\rm B} T \tag{2}$$

where $\langle X \rangle$ denotes the mean value of X. You may assume that

$$\int_{-\infty}^\infty \exp(-x^2) dx = \sqrt{\pi}.$$

Note that you will need to use integration by parts.
The solution for this question will be covered in lectures.