MTH6132, Relativity
Solutions for Problem Set 9
Due Anytime

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The Greatest Hits of the 2000s

[10 Points]
We are supplied with the relation $1/r = L\cos\varphi$ satisfied by a null geodesic $x^a(\lambda) = (t(\lambda), r(\lambda), \pi/2, \varphi(\lambda))$ with null tangent vector $\vec{u} \mapsto u^a \equiv dx^a/d\lambda$ whose norm squared is 0 along the entire geodesic

$$0 = g_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} = -f(r)\left(\frac{dt}{d\lambda}\right)^2 + \frac{1}{f(r)}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2.$$

We also know that this is a geodesic with respect to the Schwarzschild metric g_{ab}, which is a metric that is independent of both t and φ i.e. $\partial_t g_{ab} = 0$ and $\partial_\varphi g_{ab} = 0$, so we know that there are two constants E and L along the geodesic

$$\begin{aligned} -E &= g_{ta}\frac{dx^a}{d\lambda} = -f(r)\frac{dt}{d\lambda} \\ L &= g_{\varphi a}\frac{dx^a}{d\lambda} = r^2\frac{d\varphi}{d\lambda}, \end{aligned} \tag{1}$$

so

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - f(r)\frac{L^2}{r^2}.$$

We can now write down

$$\begin{aligned} \frac{dt}{dr} &= \frac{dt}{d\lambda}\frac{d\lambda}{dr} \\ &= \frac{dt}{d\lambda}\left(\frac{dr}{d\lambda}\right)^{-1} \\ &= \frac{E}{f(r)}\left(E^2 - f(r)\frac{L^2}{r^2}\right)^{-\frac{1}{2}} \\ &= \left(f(r)^2 - \frac{f(r)^3}{r^2}\frac{L^2}{E^2}\right)^{-\frac{1}{2}} \\ &= \left(f(r)^2 - r^2 f(r)\left(\frac{d\varphi}{dt}\right)^2\right)^{-\frac{1}{2}} \\ &= \left(f(r)^2 - r^2 f(r)\left(\frac{d\varphi}{dr}\frac{dr}{dt}\right)^2\right)^{-\frac{1}{2}} \\ &= \left(f(r)^2 - r^2 f(r)\left(\frac{d\varphi}{dr}\right)\left(\frac{dt}{dr}\right)^{-2}\right)^{-\frac{1}{2}}, \end{aligned}$$

so

$$\left(\frac{dt}{dr}\right)^2 = \left(f(r)^2 - r^2 f(r)\left(\frac{d\varphi}{dr}\right)\left(\frac{dt}{dr}\right)^{-2}\right)^{-1},$$

so

$$\left(\frac{dt}{dr}\right)^{-2} = f(r)^2\left(1 + r^2 f(r)\left(\frac{d\varphi}{dr}\right)\right)^{-1} = \left(\frac{1}{f(r)^2} + \frac{r^2}{f(r)}\left(\frac{d\varphi}{dr}\right)^2\right)^{-1}. \tag{2}$$

The relation $\varphi = \varphi(r)$ is supplied by the relation $1/r = L\cos\varphi$, so that $\varphi(r) = \arccos\left(\frac{1}{Lr}\right)$. This gives

$$\frac{d\varphi}{dr} = -\frac{1}{\sqrt{1-\left(\frac{1}{Lr}\right)^2}}\left(-\frac{1}{L}\frac{1}{r^2}\right) = \frac{1}{r}\frac{1}{\sqrt{L^2r^2-1}}. \tag{3}$$

We can now complete our expression for dt/dr

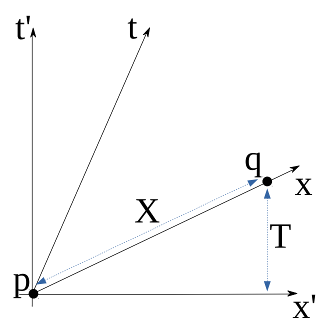
$$\frac{dt}{dr} = \left(\frac{1}{f(r)^2} + \frac{1}{f(r)(L^2r^2-1)}\right)^{\frac{1}{2}}. \tag{4}$$

Integrating this for the t interval from the point of closest approach $r = 1/L$ to $r = R$, and doubling the result to get the entire t interval from $r = R$ to $r = 1/L$ to $r = R$, we obtain the desired result

$$T = 2\int_{\frac{1}{L}}^{R}\left(\frac{1}{f(r)^2} + \frac{1}{f(r)(L^2r^2-1)}\right)^{\frac{1}{2}} dr. \tag{5}$$

The Greatest Hits of the 2010s

a) [4 Points]
We are given two points p and q that are simultaneous $\Delta t = 0$ and separated by spatial distance $\Delta x = X$ in a frame $F : (t, x)$, and has a nonzero time interval $\Delta t' = T$ and separated by some unknown spatial distance $\Delta x'$ in frame $F' : (t', x')$.



b) [3 Points]
The spatial distance $\Delta x'$ can be found by writing down the spacetime interval between points p and q both coordinate systems and equating these expressions with each other

$$-\Delta t^2 + \Delta x^2 = -\Delta t'^2 + \Delta x'^2, \tag{6}$$

so we conclude that $\Delta x' = \sqrt{\Delta x^2 + \Delta t'^2} = \sqrt{X^2 + T^2}$.

c) [3 Points]
The velocity v between frames $F : (t, x)$ and $F' : (t', x')$ can be determined by writing down the coordinates of point q in both frames and relating them by Lorentz transformation. To simplify this, let us take the point p to be at the origin. Then, the point q has coordinates $t = 0, x = X$ or $t' = T, x' = \sqrt{X^2 + T^2}$.

The Lorentz transformations at point q relates these coordinates i.e. given frame $F' : (t', x')$ moving with velocity $\beta = v/c$ with respect to $F : (t, x)$, we have

$$\begin{aligned} t &= \gamma t' + \gamma\beta x', \\ x &= \gamma\beta t' + \gamma x', \end{aligned}$$

where $\gamma = 1/\sqrt{1-\beta^2}$. Using the coordinates for point q, the first line tells us that $0 = \gamma T + \gamma\beta\sqrt{X^2+T^2}$, and thus we conclude that $v = -cT/\sqrt{X^2+T^2}$.

Gravitational Waves and Circular Rings

a) [8 Points]
Given a gravitational wave metric in coordinates $x^a = (t, x, y, z)$, with components

$$g_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cos(\omega t - kz),$$

we will now compute four different arc lengths.
The "vertical line" from the point labelled by $x = 0, y = -d/2$ and the point labelled by $x = 0, y = d/2$ which can be described by the curve $x(y) = 0, y(y) = y$ when parametrized by $\lambda = y$, has arc length

$$\sigma_{vertical} = \int_{y=-d/2}^{y=d/2} \sqrt{g_{yy}\left(\frac{dy}{dy}\right)^2}\, dy = d\sqrt{1 - h_+\cos(\omega t)}.$$

The "horizontal line" from the point labelled by $x = -d/2, y = 0$ and the point labelled by $x = d/2, y = 0$ which can be described by $x(x) = x, y(x) = 0$ when parametrized by $\lambda = x$, has arc length

$$\sigma_{horizontal} = \int_{x=-d/2}^{x=d/2} \sqrt{g_{xx}\left(\frac{dx}{dx}\right)^2}\, dy = d\sqrt{1 + h_+\cos(\omega t)}.$$

The "upward sloping diagonal" from the point labelled by $x = -d/(2\sqrt{2}), y = -d/(2\sqrt{2})$ and the point labelled by $x = d/(2\sqrt{2}), y = d/(2\sqrt{2})$ which can be described by $x(x) = x, y(x) = x$ when parametrized by $\lambda = x$, has arc length

$$\sigma_{upward} = \int_{x=-d/(2\sqrt{2})}^{x=d/(2\sqrt{2})} \sqrt{g_{xx}\left(\frac{dx}{dx}\right)^2 + 2g_{xy}\left(\frac{dx}{dx}\right)\left(\frac{dy}{dx}\right) + g_{yy}\left(\frac{dy}{dx}\right)^2}\, dy = d\sqrt{1 + h_\times\cos(\omega t)}.$$

The "downward sloping diagonal" from the point labelled by $x = -d/(2\sqrt{2}), y = d/(2\sqrt{2})$ and the point labelled by $x = d/(2\sqrt{2}), y = -d/(2\sqrt{2})$ which can be described by $x(x) = x, y(x) = -x$ when parametrized by $\lambda = x$, has arc length

$$\sigma_{downward} = \int_{x=-d/(2\sqrt{2})}^{x=d/(2\sqrt{2})} \sqrt{g_{xx}\left(\frac{dx}{dx}\right)^2 + 2g_{xy}\left(\frac{dx}{dx}\right)\left(\frac{dy}{dx}\right) + g_{yy}\left(\frac{dy}{dx}\right)^2}\, dy = d\sqrt{1 - h_\times\cos(\omega t)}.$$

b) [2 Points]
The effect of the gravitational wave on the circular ring is to alternately stretch and squeeze it along the x and y axes with amplitude h_+, and along the diagonal axes offset by $\pi/4$ from the x and y axes with amplitude $h_\times$.