MTH6132, Relativity

Solutions for Problem Set 8

Due 12$^{\text{th}}$ December 2018

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The Photon Sphere

a) [5 Points]

Any geodesic $x^a(\lambda) = (t(\lambda), r(\lambda), \pi/2, \varphi(\lambda))$ in Schwarzschild where $f(r) = 1 - 2GM/r$ has two constants

$$\begin{aligned} -E &= -f(r)\frac{dt}{d\lambda} \\ L &= r^2\frac{d\varphi}{d\lambda}, \end{aligned}$$

and satisfy

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - U(r).$$

Taking a derivative with respect to λ of the left-hand side gives $2(dr/d\lambda)(d^2r/d\lambda^2)$, and of the right-hand side gives $-dU/d\lambda = -(dr/d\lambda)(dU/dr)$, which gives

$$\frac{d^2r}{d\lambda^2} = -\frac{1}{2}\frac{dU}{dr}.$$

b) [5 Points]

For a null geodesic, $U(r) \equiv f(r)\frac{L^2}{r^2}$ where $f(r) = 1 - 2GM/r$. Let us write this down explicitly

$$U(r) = \frac{L^2}{r^2} - \frac{2GML^2}{r^3},$$

so

$$\frac{dU}{dr} = -\frac{2L^2}{r^3} + \frac{6GML^2}{r^4}.$$

A circular geodesic at some constant $r = R$ satisfies both $(dr/d\lambda)|_{r=R} = 0$ and $(d^2r/d\lambda^2)|_{r=R} = 0$. The first of these gives a relation between the constants R, E, L, and the second of these for a null geodesic with $dU/dr = -2L^2/r^3 + 6GML^2/r^4$ gives $R = 3GM$.

Orbital Period

a) [5 Points]
For a timelike geodesic, $U(r) \equiv f(r)(1+\frac{L^2}{r^2})$ where $f(r) = 1 - 2GM/r$. Let us write this down explicitly

$$U(r) = 1 - \frac{2GM}{r} + \frac{L^2}{r^2} - \frac{2GML^2}{r^3},$$

so

$$\frac{dU}{dr} = \frac{2GM}{r^2} - \frac{2L^2}{r^3} + \frac{6GML^2}{r^4}.$$

A circular geodesic at some constant $r = R$ satisfies both $(dr/d\lambda)|_{r=R} = 0$ and $(d^2r/d\lambda^2)|_{r=R} = 0$. The first of these gives a relation between the constants R, E, L, and the second of these for a timelike geodesic with $dU/dr = 2GM/r^2 - 2L^2/r^3 + 6GML^2/r^4$ gives $R = (L^2/2GM)\left(1 \pm \sqrt{1 - 12G^2M^2/L^2}\right)$.

b) [10 Points]
For a geodesic $x^a(\lambda) = (t(\lambda), r(\lambda), \pi/2, \varphi(\lambda))$, define the period of its orbit T by $d\varphi/dt = 2\pi/T$. Let us now specialize to a circular geodesic so $r = R$, and one that is timelike so $U(r) \equiv f(r)(1+\frac{L^2}{r^2})$. To find T, it suffices to find $d\varphi/dt$, which we now write down explicitly at $r = R$

$$\frac{d\varphi}{dt} = \frac{d\varphi}{d\lambda}\frac{d\lambda}{dt} = \frac{d\varphi}{d\lambda}\left(\frac{dt}{d\lambda}\right)^{-1} = \left(\frac{L}{R^2}\right)\left(\frac{E}{f(R)}\right)^{-1} = \frac{f(R)}{R^2}\frac{L}{E} = \frac{(R-2GM)}{R^3}\frac{L}{E}.$$

We now solve for L and E, in that order, in terms of R. The condition $(d^2r/d\lambda^2)|_{r=R} = 0$ satisfied by a circular geodesic

$$0 = \left.\frac{dU}{dr}\right|_{r=R} = \frac{2GM}{R^2} - \frac{2L^2}{R^3} + \frac{6GML^2}{R^4},$$

gives

$$L^2 = \frac{GMR^2}{(R-3GM)}.$$

The other condition $(dr/d\lambda)|_{r=R} = 0$ satisfied by a circular geodesic

$$0 = E^2 - U(R) = E^2 - f(R)\left(1 + \frac{L^2}{R^2}\right) = E^2 - \left(\frac{R-2GM}{R}\right)\left(1 + \frac{GM}{R-3GM}\right),$$

gives

$$E^2 = \frac{(R-2GM)^2}{R(R-3GM)}.$$

Putting everything together,

$$\begin{aligned}\frac{d\varphi}{dt} &= \frac{(R-2GM)}{R^3}\frac{L}{E} \\ &= \frac{(R-2GM)}{R^3}\sqrt{\left(\frac{GMR^2}{(R-3GM)}\right)\left(\frac{(R-2GM)^2}{R(R-3GM)}\right)^{-1}} \\ &= \sqrt{\frac{GM}{R^3}},\end{aligned} \tag{1}$$

so $T = 2\pi/(d\varphi/dt) = 2\pi\sqrt{R^3/GM}$.

Radially Infalling Observer in Schwarzschild

a) [1 Point]
We are to consider an observer moving along a timelike geodesic $x^a(\tau) = (t(\tau), r(\tau), \pi/2, \varphi(\tau))$, and thus has $U(r) \equiv f(r)(1 + \frac{L^2}{r^2})$ with

$$\begin{aligned}\left(\frac{dr}{d\tau}\right)^2 &= E^2 - U(r) \\ &= E^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{L^2}{r^2}\right).\end{aligned}$$

b) [1 Point]
For a radially infalling observer, we have $d\varphi/d\tau = 0$ and thus $L = r^2(d\varphi/d\tau) = 0$. For an observer that starts from rest at some finite $r(0) = R$, we have $(dr/d\tau)|_{r=R} = 0$ and thus $E^2 = (1 - \frac{2GM}{R})$.

c) [1 Point]
We can straightforwardly rewrite the differential equation relating the radial coordinate r and the proper time τ in cycloid form

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{\left(\frac{R}{2}\right)^2}{\left(\left(\frac{R}{2}\right)\left(\frac{R_s}{R}\right)^{-\frac{1}{2}}\right)^2}\left(\frac{2\left(\frac{R}{2}\right) - r}{r}\right).$$

where $R_s = 2GM$ is the Schwarzschild radius. We have undertaken this effort to make use of the known parametrization of the cycloid

$$\begin{cases} x(\eta) = a\,(\eta \pm \sin\eta) \\ y(\eta) = b\,(1 \pm \cos\eta) \end{cases}$$

which satisfies the differential equation

$$\left(\frac{dy}{dx}\right)^2 = \frac{b^2}{a^2}\left(\frac{2b - y}{y}\right).$$

d) [1 Point]

By direct comparison, and by choosing a parametrization such that we recover $r(0) = R$ (this chooses the "+" branch), we have an explicit relation between r and τ for all points along the radially infalling particle's geodesic curve

$$\begin{cases} \tau(\eta) = \left(\frac{R}{2}\right)\left(\frac{R_s}{R}\right)^{-\frac{1}{2}}(\eta + \sin\eta) \\ r(\eta) = \frac{R}{2}(1 + \cos\eta) \end{cases}$$

where the parameter $\eta \in [0, \pi]$ is such that $\tau(0) = 0$, $r(0) = R$, and $\tau(\pi) = \left(\frac{R}{2}\right)\left(\frac{R_s}{R}\right)^{-\frac{1}{2}}(\pi + \sin\pi)$, $r(\pi) = 0$.

e) [1 Point]

We are to compute the proper time experienced by a radially infalling observer from its initial radius $R = R_s = 2GM$ to its final radius $r = 0$. Given our explicit parametrization of the entire geodesic that such an observer travels along, we immediately see that

$$\tau(\pi) = \left(\frac{R_s}{2}\right)\left(\frac{R_s}{R_s}\right)^{-\frac{1}{2}}(\pi + \sin\pi) = \frac{\pi}{2}R_s = \pi GM$$

which occurs for $E^2 = \left(1 - \frac{R_s}{R_s}\right) = 0$.