MTH6132, Relativity

Problem Set 8

Due 12$^{\text{th}}$ December 2018

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Let us investigate timelike and null geodesics $x^a(\lambda) = (t(\lambda), r(\lambda), \pi/2, \varphi(\lambda))$ with respect to Schwarzschild

$$g_{ab}dx^a dx^b = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

$$f(r) = 1 - 2GM/r.$$

with two constant along any geodesic from the fact that $\partial_t g_{ab} = 0$ and $\partial_\varphi g_{ab} = 0$ which give

$$\begin{aligned} -E &= -f(r)\frac{dt}{d\lambda} \\ L &= r^2\frac{d\varphi}{d\lambda}. \end{aligned}$$

The Photon Sphere

A geodesic is *circular* if $r = const$ along the entire geodesic. The Schwarzschild geometry admits null circular geodesics at a surface known as the *photon sphere*. For a null geodesic, the first and second geodesic motion equations are

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - U(r), \tag{1}$$

$$\frac{d^2r}{d\lambda^2} = -\frac{1}{2}\frac{dU}{dr}, \tag{2}$$

where $U(r)$ for null geodesics is

$$U(r) \equiv f(r)\frac{L^2}{r^2}.$$

a) Show how one obtains (2) from (1).

b) Show that the photon sphere for Schwarzschild is at $r = 3GM$.

Orbital Period

The Schwarzschild geometry also admits timelike circular geodesics. For a timelike geodesic parametrized by proper time which we will denote as λ for now, the first and second geodesic motion equations are (1) and (2), respectively, where $U(r)$ for timelike geodesics is

$$U(r) \equiv f(r)\left(1 + \frac{L^2}{r^2}\right).$$

a) Show that timelike circular geodesics are possible at $r = \frac{L^2}{2GM}\left(1 \pm \sqrt{1 - \frac{12G^2M^2}{L^2}}\right)$.

b) Show that for a timelike circular geodesic at $r(\lambda) = R$, the period of the orbit T defined by $d\varphi/dt = 2\pi/T$, i.e. measured in Schwarzschild time t which is the proper time of the observer at $r \to \infty$,

$$T = 2\pi\sqrt{\frac{R^3}{GM}}.$$

Radially Infalling Observer in Schwarzschild

An observer has a worldline $x^a(\tau) = (t(\tau), r(\tau), \pi/2, \varphi(\tau))$ parametrized by proper time τ that is a timelike geodesic with respect to the Schwarzschild metric.

a) Show that

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{L^2}{r^2}\right) \tag{3}$$

(it is amusing to see that the observer that struggles the least, namely the one that isn't wiggling in the φ direction so that $L = 0$, actually has a smaller $dr/d\tau$ and so experiences more proper time before reaching $r = 0$).

A geodesic is *radially infalling* if $\varphi = const$ along the entire geodesic

b) Show that for a radially infalling observer who starts from rest $(dr/d\tau)|_{r=R} = 0$ at some finite $r(0) = R$, the values of the constants E, L are fixed to be

$$\begin{aligned} E^2 &= \left(1 - \frac{2GM}{R}\right) \\ L &= 0. \end{aligned} \tag{4}$$

A *cycloid* is a curve $x(\eta), y(\eta)$ for $\eta \in [0, \pi]$ given by

$$\begin{cases} x(\eta) = a\,(\eta \pm \sin\eta) \\ y(\eta) = b\,(1 \pm \cos\eta) \end{cases} \tag{5}$$

which is the solution to the *cycloid differential equation*

$$\left(\frac{dy}{dx}\right)^2 = \frac{b^2}{a^2}\left(\frac{2b - y}{y}\right). \tag{6}$$

c) Using (4), show that for a radially infalling observer we can rewrite (3) in cycloid form

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{\left(\frac{R}{2}\right)^2}{\left(\left(\frac{R}{2}\right)\left(\frac{2GM}{R}\right)^{-\frac{1}{2}}\right)^2}\left(\frac{2\left(\frac{R}{2}\right) - r}{r}\right). \tag{7}$$

d) Compare (7) to the cycloid differential equation (6), and show how you can use the known solution (5) of this differential equation to conclude that

$$\begin{cases} \tau(\eta) = \left(\frac{R}{2}\right)\left(\frac{2GM}{R}\right)^{-\frac{1}{2}}(\eta + \sin\eta) \\ r(\eta) = \frac{R}{2}\,(1 + \cos\eta) \end{cases}$$

where the parameter $\eta \in [0, \pi]$ labels the point where the observer starts from rest $\tau(0) = 0$, $r(0) = R$ by $\eta = 0$, and the point where the observer's worldline ends $\tau(\pi) = \tau_{max}$, $r(\pi) = 0$ by $\eta = \pi$.

e) Thus, conclude that the proper time experiences by a radially infalling observer in Schwarzschild, from an initial radius of $R = 2GM$ to a final radius of $r = 0$, is finite and is given by

$$\tau(\pi) - \tau(0) = \pi GM. \tag{8}$$