MTH6132, Relativity

Solutions to Problem Set 7

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1. [9 points] Identify coordinates $x^1 = x, x^2 = y$.

Metric and its inverse: $g_{ab} = \begin{pmatrix} e^y & 0 \\ 0 & e^x \end{pmatrix}$, $g^{ab} = \begin{pmatrix} e^{-y} & 0 \\ 0 & e^{-x} \end{pmatrix}$

(a) The connection coefficients are $\Gamma^a{}_{bc} = \frac{g^{ad}}{2}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right)$

$\Gamma^1{}_{11} = \frac{1}{2}g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) = 0 \quad$ since $\partial_1 g_{11} = \partial_x e^y = 0$

$\Gamma^1{}_{12} = \frac{1}{2}g^{11}(\partial_2 g_{11} + \partial_1 g_{12} - \partial_1 g_{12}) = \frac{1}{2} \quad$ since $\partial_2 g_{11} = \partial_y e^y = e^y,\ g^{11} = e^{-y},\ g_{12} = 0$

$\Gamma^1{}_{22} = \frac{1}{2}g^{11}(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}) = -\frac{e^{x-y}}{2} \quad$ since $\partial_1 g_{22} = \partial_x e^x = e^x,\ g^{11} = e^{-y},\ g_{12} = 0$

$\Gamma^2{}_{11} = \frac{1}{2}g^{22}(\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11}) = -\frac{e^{y-x}}{2} \quad$ since $\partial_2 g_{11} = \partial_y e^y = e^y,\ g^{22} = e^{-x},\ g_{12} = 0$

$\Gamma^2{}_{12} = \frac{1}{2}g^{22}(\partial_2 g_{21} + \partial_1 g_{22} - \partial_2 g_{12}) = \frac{1}{2} \quad$ since $\partial_1 g_{22} = \partial_x e^x = e^x,\ g^{22} = e^{-x},\ g_{12} = 0$

$\Gamma^2{}_{22} = \frac{1}{2}g^{22}(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}) = 0 \quad$ since $\partial_2 g_{22} = 0$

(b) The Riemann Tensor is $R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}$.

$$\begin{aligned}
R^1{}_{212} &= \partial_1 \Gamma^1{}_{22} - \partial_2 \Gamma^1{}_{21} + \Gamma^1{}_{e1}\Gamma^e{}_{22} - \Gamma^1{}_{e2}\Gamma^e{}_{21} \\
&= \partial_1 \Gamma^1{}_{22} - \partial_2 \Gamma^1{}_{21} + \Gamma^1{}_{11}\Gamma^1{}_{22} + \Gamma^1{}_{21}\Gamma^2{}_{22} - \Gamma^1{}_{12}\Gamma^1{}_{21} - \Gamma^1{}_{22}\Gamma^2{}_{21} \\
&= -\frac{e^{x-y}}{2} - \frac{1}{4} + \frac{e^{x-y}}{4} \\
&= -\frac{1}{4} - \frac{e^{x-y}}{4} \\
&= -\frac{e^y + e^x}{4\,e^y}.
\end{aligned}$$

Finally,

$$\begin{aligned}
R_{1212} &= g_{1a}R^a{}_{212} \\
&= g_{11}R^1{}_{212} + g_{12}R^2{}_{212} \\
&= e^y\left(-\frac{e^y + e^x}{4\,e^y}.\right) \\
&= -\frac{e^y + e^x}{4}.
\end{aligned}$$

2. [9 points] Using the formula for the Ricci tensor one has that

$$\begin{aligned}
R_{tt} = R_{00} &= \partial_a \Gamma^a{}_{00} - \partial_0 \Gamma^a{}_{0a} + \Gamma^a{}_{ea}\Gamma^e{}_{00} - \Gamma^a{}_{e0}\Gamma^e{}_{0a}, \\
&= \partial_0 \Gamma^0{}_{00} + \partial_1 \Gamma^1{}_{00} - \partial_0 \Gamma^0{}_{00} - \partial_0 \Gamma^1{}_{01} \\
&\qquad + \Gamma^0{}_{e0}\Gamma^e{}_{00} + \Gamma^1{}_{e1}\Gamma^e{}_{00} - \Gamma^0{}_{e0}\Gamma^e{}_{00} - \Gamma^1{}_{e0}\Gamma^e{}_{01}, \\
&= \partial_1 \Gamma^1{}_{00} + \Gamma^0{}_{e0}\Gamma^e{}_{00} + \Gamma^1{}_{e1}\Gamma^e{}_{00} - \Gamma^0{}_{e0}\Gamma^e{}_{00} - \Gamma^1{}_{e0}\Gamma^e{}_{01}, \\
&= \partial_1 \Gamma^1{}_{00} + \Gamma^0{}_{00}\Gamma^0{}_{00} + \Gamma^0{}_{10}\Gamma^1{}_{00} + \Gamma^1{}_{01}\Gamma^0{}_{00} + \Gamma^1{}_{11}\Gamma^1{}_{00} \\
&\qquad - \Gamma^0{}_{00}\Gamma^0{}_{00} - \Gamma^0{}_{10}\Gamma^1{}_{00} - \Gamma^1{}_{00}\Gamma^0{}_{01} - \Gamma^1{}_{10}\Gamma^1_{01}, \\
&= \partial_1 \Gamma^1{}_{00} + \Gamma^0{}_{10}\Gamma^1{}_{00} - \Gamma^0{}_{10}\Gamma^1{}_{00} - \Gamma^1{}_{00}\Gamma^0{}_{01}, \\
&= A\partial_r\left(e^{2Ar}\right) - A^2 e^{2Ar} = A^2 e^{2Ar}.
\end{aligned}$$

3. [6 points] (a) To compute $\nabla_e R_{abcd}$ one uses the Leibniz rule:

$$\nabla_e R_{abcd} = K\left(\cancelto{0}{(\nabla_e g_{ac})} g_{bd} + g_{ac} \cancelto{0}{(\nabla_e g_{bd})} - \cancelto{0}{(\nabla_e g_{ac})} g_{bd} - g_{ac} \cancelto{0}{(\nabla_e g_{bd})} \right) = 0.$$

(b) To compute R_{bd} proceed as follows:

$$\begin{aligned} R_{bd} &= g^{ac} R_{abcd} = K g^{ac}(g_{ac}g_{bd} - g_{ad}g_{cb}) = K(g^{ac}g_{ac}g_{bd} - g^{ac}g_{ad}g_{cb}) \\ &= K(\delta_a{}^a g_{bd} - \delta_d{}^c g_{cb}) \\ &= K(4g_{bd} - g_{bd}) = 3K g_{bd}, \end{aligned}$$

where it has been used that $\delta_a{}^a = 4$. Finally

$$R = g^{bd} R_{bd} = 3K g^{bd} g_{bd} = 12K.$$

4. [6 points] From

$$R_{ab} - \frac{1}{2} R g_{ab} + \lambda g_{ab} = 0, \tag{1}$$

multiplying by g^{ab} one obtains that

$$g^{ab} R_{ab} - \frac{1}{2} R g^{ab} g_{ab} + \lambda g^{ab} g_{ab} = 0.$$

Now, by definition $R = g^{ab} R_{ab}$. Moreover, as seen in a previous coursework one has that $g^{ab} g_{ab} = 4$. Thus, one has that

$$R - 2R + 4\lambda = 0$$

from where one readily obtains $R = 4\lambda$. Substituting this result into (1) and simplifying one obtains $R_{ab} = \lambda g_{ab}$ as required.