MTH6132, Relativity

Problem Set 4
Due 31th October 2018

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Metric: coordinate change and matrix representation

1. Starting from the Minkowski line element in Cartesian coordinates

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2,$$

show that in spherical coordinates $x = r\sin\theta\sin\varphi$, $y = r\sin\theta\cos\varphi$, $z = r\cos\theta$, the line element is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2.$$

1. Starting from the line element of a black hole in spherical coordinates (t, r, θ, φ)

$$\mathrm{d}s^2 = -F(r)\mathrm{d}t^2 + \frac{1}{F(r)}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin\theta^2\mathrm{d}\varphi^2, \quad \text{with} \quad F(r) = 1 - \frac{r_0}{r}, \quad r_0 \geq 0$$

show that in the coordinates $\{u, R, \Theta, \phi\}$ given by

$$\begin{aligned} t &= u + r^*(R), \quad \text{with} \quad r^*(R) = R + r_0 \ln\left(\frac{R}{r_0} - 1\right) \\ r &= R, \quad \theta = \Theta, \quad \varphi = \phi \end{aligned}$$

the line element is [hint: notice that $\frac{dr^*}{dR} = \frac{1}{F(R)}$]

$$\mathrm{d}s^2 = -F(R)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}R + R^2\mathrm{d}\Theta^2 + R^2\sin^2\Theta\mathrm{d}\phi^2. \tag{1}$$

3. The Bondi metric, used in the study of gravitational radiation, has line element in the coordinate (u, R, Θ, ϕ) given by [f, g, α and β are functions of (u, R, Θ, ϕ)]

$$\begin{aligned} ds^2 &= -\left(\frac{f}{R}e^{2\beta} - g^2R^2e^{2\alpha}\right)du^2 - 2e^{2\beta}dudR - 2gR^2e^{2\alpha}du\,d\theta \\ &\quad + R^2\left(e^{2\alpha}d\Theta^2 + e^{-2\alpha}\sin^2\Theta d\phi^2\right). \end{aligned} \tag{2}$$

(a) Using these coordinates, write down the matrix representation of the metric g_{ab} [hint: first write down x^a].

(b) For the particular case of the metric given in exercise 2, write down the functions f, g, α and β [hint: compare eqs. (1) and (2)]

(c) Write down the matrix representation of the inverse metric g^{ab} for the particular case of item 3(b).

Please turn

Metric: covariant/contravariant tensors and scalar products

4. Start with spherical coordinates (r, θ, φ) in $\mathbb{R}^3$ and the line element given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2.$$

(a) Given the contravariant vector $X^a = (1, r, r^2)$, find X_a.
(b) Given the covariant vector $Y_a = (0, -r^2, r^2\cos^2\theta))$, find Y^a.

5. Consider $\mathbb{R}^2$ with the standard Euclidean metric in polar coordinates $ds^2 = dr^2 + r^2 d\theta^2$. Suppose $V^a = (1, 1)$ and $W_a = (0, 2)$. Compute the inner product of V and W in the following two ways:

$$\begin{aligned} V \cdot W &= g_{ab} V^a W^b \\ V \cdot W &= V^a W_a. \end{aligned}$$

Do your answers agree? Why or why not?