MTH6132, Relativity

Solutions for Problem Set 1

Due 10$^{\text{th}}$ October 2018

Hans Bantilan

The Barn and Pole Paradox

The barn (blue) with proper length L is at rest in frame $F : (t, x, y, z)$, and the pole (red) is at rest in frame $F' : (t', x', y', z')$. Measured in frame F, the length of the pole is L. Frames F and F' are in standard configuration with boost parameter β.

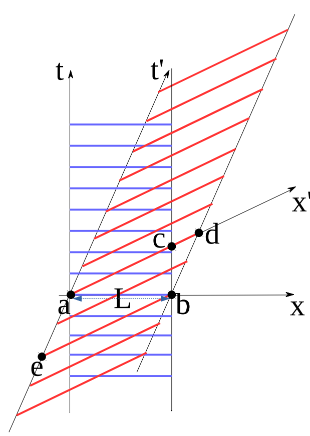
In the following, whenever we write down an interval, e.g. Δx, we shall specify its endpoints by subscripts, e.g. $\Delta x_{pq} = |x_p - x_q|$ is the absolute value of the difference in x coordinate values of point p and point q.

We will also make frequent use of the Lorentz transformations at some point p for frames $F : (t, x, y, z)$ and $F' : (t', x', y', z')$ in standard configuration with boost parameter β

$$
\begin{aligned}
t &= \gamma t' + \gamma\beta x', \\
x &= \gamma\beta t' + \gamma x', \\
y &= y', \\
z &= z', \qquad (1)
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
t' &= \gamma t - \gamma\beta x, \\
x' &= -\gamma\beta t + \gamma x, \\
y &= y', \\
z &= z'. \qquad (2)
\end{aligned}
$$



a) [4 Points]

We are to compute the length of the barn $\Delta x'_{ac}$ measured in frame F'. First, some intuition: since this length measured in a frame that is not the rest frame of the barn, we expect this length to be contracted compared to the proper length L of the barn, i.e. $\Delta x'_{ac} < L$.

To verify these expectations, let us compute. Consider the point c. It can be labeled as $(t, x) = (\Delta t_{bc}, L)$ or as $(t', x') = (0, \Delta x'_{ac})$. Substituting these values into the Lorentz transformations (1) at point c

$$
\begin{aligned}
\cancelto{\Delta t_{bc}}{t} &= \gamma \cancelto{0}{t'} + \gamma\beta \cancelto{\Delta x'_{ac}}{x'} \\
\cancelto{L}{x} &= \gamma\beta \cancelto{0}{t'} + \gamma \cancelto{\Delta x'_{ac}}{x'},
\end{aligned}
$$

immediately tells us that $\Delta x'_{ac} = L/\gamma$.

b) [4 Points]

We are to compute the length of the pole $\Delta x'_{ad}$ measured in frame F' i.e. its proper length. First, some intuition: since this is the proper length of the pole, we expect it to be greater than its length measured in any other frame e.g. since the pole has length L in frame F which is not the rest frame of the pole, we expect $\Delta x'_{ad} > L$.

To verify these expectations, let us compute. Consider the point[1] b. It can be labeled as $(t, x) = (0, L)$ or as $(t', x') = (-\Delta t'_{bd}, \Delta x'_{ad})$. Substituting these values into the Lorentz transformations (2) at point b

$$
\begin{aligned}
\cancelto{-\Delta t'_{bd}}{t'} &= \gamma \cancelto{0}{t} - \gamma\beta \cancelto{L}{x} \\
\cancelto{\Delta x'_{ad}}{x'} &= -\gamma\beta \cancelto{0}{t} + \gamma \cancelto{L}{x},
\end{aligned}
$$

immediately tells us that $\Delta x'_{ad} = \gamma L$.

c) [5 Points]

We are to draw the situation of a runner holding the pole in the x direction, running through the barn whose front and rear doors are open, with a boost parameter β with respect to the F frame. This situation is depicted in the spacetime diagram above.

d) [1 Point]

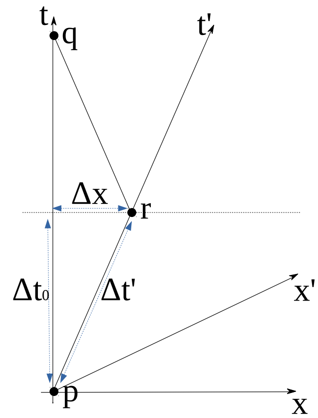
In frame F, the situation can be stated as: "the pole fits in the barn between points a and b at $t = 0$".

In frame F', the situation can be stated as: "1) when the pole's front end is at point b, its back end is at point e i.e. at $t' = -\Delta t'_{ae}$, its back end sticking out of the barn, 2) when the pole's back end is point a, its front end is at point d i.e. at $t' = 0$, its front end is sticking out of the barn".

[1] We can also solve this problem by considering the point d, but this will require a lengthier calculation.

The Twin Paradox

The Sol and Alpha Centauri systems are both at rest in frame $F : (t, x, y, z)$. Measured in frame F, the Alpha Centauri system is Δx away from the Sol system. A ship at rest in frame $F' : (t', x', y', z')$ travels from Sol to Alpha Centauri. Frames F and F' are in standard configuration with boost parameter β.



a) [1 Point]
We are to compute the distance $\Delta x'_{pr}$ measured in frame F'.

Since the ship labels both point p and point q with $x' = 0$, we see that $\Delta x'_{pr} = 0$.

b) [4 Points]
We are to compute the elapsed time $\Delta t'$ (see diagram) measured in frame F' in terms of Δx (see diagram). Consider the point r. It can be labeled as (see diagram) $(t, x) = (\Delta t_0, \Delta x)$ or as $(t', x') = (\Delta t', 0)$. Substituting these values into the Lorentz transformations (1) at point r

$$\begin{aligned} \cancelto{\Delta t_0}{t} &= \gamma \cancelto{\Delta t'}{t'} + \gamma\beta \cancelto{0}{x'} \\ \cancelto{\Delta x}{x} &= \gamma\beta \cancelto{\Delta t'}{t'} + \gamma \cancelto{0}{x'}, \end{aligned}$$

immediately tells us that $\Delta t' = \Delta x/(\gamma\beta)$.

c) [4 Points]
We are to compute the elapsed time Δt_0 (see diagram) as measured in frame F frame in terms of Δx (see diagram). We can use the calculation in part (b) to immediately see that $\Delta t_0 = \gamma \Delta t' = \gamma\left(\Delta x/(\gamma\beta)\right) = \Delta x/\beta$.

d) [5 Points]
The ship travels from the Sol system to the Alpha Centauri system at constant boost β, and as soon as it arrives it immediately travels from the Alpha Centauri system to the Sol system at the same constant boost β. The two trajectories of the twins have the same endpoints: label these endpoints p and q. This situation is depicted in the spacetime diagram above.

e) [2 Points]
The twin paradox can be stated as follows. Twin A stays in the Sol system while twin B travels to the Alpha Centauri system with the trajectory described in (d). Twin A expects that the other has experienced more elapsed time, while twin B also expects that the other has experienced more elapsed time.

Twin A experiences $2\Delta t_0 = 2\Delta x/\beta$ of proper time, while twin B experiences $2\Delta t' = 2\Delta x/(\gamma\beta)$ of proper time. Twins A and B are of the same age at point p, but by the time they meet again at point q twin A is older than twin B.