Main Examination period 2018

MTH6132: Relativity

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: H. T. Nguyen, J. A. Valiente-Kroon

Question 1. [15 marks] Let $\overline{A}, \overline{B}$ denote two arbitrary 4-vectors in Minkowski space-time.

(a) Define the scalar product $\overline{A} \cdot \overline{B}$. **[3]**

(b) State the definition for the **invariance** of $\overline{A} \cdot \overline{B}$. **[2]**

(c) Show that if $|\overline{A}|^2, |\overline{B}|^2$ and $|\overline{A} + \overline{B}|^2$ are invariant, then $\overline{A} \cdot \overline{B}$ is invariant. **[3]**

(d) Define **spacelike**, **timelike** and **null** vectors in Minkowski space. **[4]**

(e) Show that the sum of any two orthogonal spacelike vectors is also spacelike. **[3]**

Question 2. [16 marks]

(a) Let ϕ be a scalar and let $V_a = \frac{\partial \phi}{\partial x^a} = \partial_a \phi$. Show that V_a is tensorial. **[6]**

(b) Consider $\mathbb{R}^3$ with spherical co-ordinates (r, θ, φ) and line element given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2.$$

(i) Given the contravariant vector $X^a = (1, r, r^2)$, find X_a. **[5]**

(ii) Given the covariant vector $Y_a = (0, -r^2, r^2 \cos^2 \theta)$, find Y^a. **[5]**

Question 3. [16 marks] Let F and F' denote two inertial reference frames moving with velocity v with respect to each other.

(a) Given the Lorentz transformations in the appendix, compute the inverse Lorentz transformations. **[4]**

(b) Let $\Delta = -c^2 t^2 + x^2 + y^2 + z^2$. Show that Δ is invariant under Lorentz transformations. **[4]**

(c) Consider a particle moving along the x-axis. Its velocity in the x direction with respect to the frames F and F' are given, respectively, by

$$V = \frac{dx}{dt} \quad \text{and} \quad V' = \frac{dx'}{dt'}.$$

(i) Using the Lorentz transformation between F and F' show that

$$V' = \frac{V - v}{1 - Vv/c^2}.$$

[6]

(ii) State a formula for V in terms of V'. **[2]**

Question 4. [18 marks] Consider a space-time with metric

$$ds^2 = -e^{2Ar}dt^2 + dr^2.$$

(a) Write down the tensors g_{ab} and g^{ab} corresponding to this metric. **[4]**

(b) Compute the Lagrangian, L, of this metric. **[4]**

(c) Compute the Christoffel symbols and geodesic equations for this metric. **[10]**

Question 5. [15 marks]

(a) State the definition of the momentum 4-vector and the law of conservation of momentum. **[5]**

(b) In this question, assume that we are using units for which $c = 1$. A particle has rest mass m_0. While at rest, it emits a photon and its rest mass is reduced to $m_0/2$.

(i) Show that the speed of the particle after the reduction of mass is $3/5$. **[8]**

(ii) Compute the value of the energy of the photon, $E = h\nu$, in terms of m_0. **[2]**

Question 6. [20 marks] Consider the Schwarzschild metric, given in local co-ordinates (t, r, θ, φ) by

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2GM}{r}\right)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1}$$

(a) If $A = 1 - \frac{2GM}{r}$ and $A' = \frac{2GM}{r^2}$ derive the geodesic equations for the metric in the form above (1) in terms of A, A'. **[8]**

(b) Describe which metric this is when $M = 0$. **[2]**

(c) Find the values of r for which this metric is singular. **[4]**

(d) Consider the following new co-ordinate system

$$\hat{t} = t + 2GM \ln|r - 2GM|, r = r, \theta = \theta, \varphi = \varphi.$$

Compute the line element in these co-ordinates. **[6]**

End of Paper – An appendix of 1 page follows.

- Lower case Latin indices run from 0 to 3
- The metric tensor of Minkowski space-time is η_{ab} where
$$ds^2 = \eta_{ab} dx^a dx^b = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$
- The Lorentz transformations between two frames F and F' in standard configuration are given by
$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad y' = y, \quad z' = z, \quad \text{with } \gamma = \frac{1}{\sqrt{1 - (v^2/c^2)}}$$
where F' is moving with speed v relative to F.
- A covariant vector is tensorial if
$$V'_a = \frac{\partial x^b}{\partial x'^a} V_b$$
and a contravariant vector is tensorial if
$$V'^a = \frac{\partial x'^a}{\partial x^b} V^b$$
- The covariant derivative of a covariant vector is given by
$$\nabla_a V_b = \partial_a V_b - \Gamma^c{}_{ab} V_c.$$
- The covariant derivative of a contravariant vector is given by
$$\nabla_a V^b = \partial_a V^b + \Gamma^b{}_{ac} V^c.$$
- The metric tensor satisfies
$$g_{ab} g^{bc} = \delta^c_a.$$
- The Christoffel symbols (connection):
$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{da} - \partial_d g_{ab})$$
- The Riemann curvature tensor :
$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ec} \Gamma^e{}_{bd} - \Gamma^a{}_{ed} \Gamma^e{}_{bc}$$
- Euler-Lagrange Equations:
$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) - \frac{\partial L}{\partial x^c} = 0.$$
- Geodesic equations:
$$\ddot{x}^a + \Gamma^a{}_{bc} \dot{x}^b \dot{x}^c = 0.$$

End of Appendix.