Week 8: Cosmic Microwave Background anisotropies

Please hand in the completed problems by **Wednesday 20th of November at 4pm**. *Please show your working and write neatly, staple all sheets together, and write your name and student number at the top of the first sheet.*

1. **Maths practice: Spherical harmonics**
   (a) What are the possible values of m for the monopole ($\ell = 0$), dipole ($\ell = 1$), and hexadecapole ($\ell = 16$)?
   (b) Sketch the monopole and dipole modes on a Mollweide projection (choose any direction for the dipole).
   (c) Show that the monopole and dipole modes are *orthogonal*, i.e. that
   $$\int_0^\pi \int_0^{2\pi} Y_{0,0}(\theta, \phi) Y^*_{1,m}(\theta, \phi) \sin\theta \, d\theta d\phi = 0,$$
   where * denotes the complex conjugate, and this expression should hold for any value of m. You can use the following definitions for the $Y_{\ell,m}$ functions:
   $$Y_{0,0} \propto 1; \qquad Y_{1,0} \propto \cos\theta; \qquad Y_{1,\pm1} \propto \sin\theta \, e^{im\phi}.$$

2. **Angular scales**
   (a) Calculate the approx. angular scales corresponding to (i) $\ell = 10$; (ii) $\ell = 100$; (iii) $\ell = 1000$, in degrees.
   (b) A typical galaxy cluster has a physical size of about 1 Mpc. For a galaxy cluster at an angular diameter distance of 4.2 Gpc from us, what value of ℓ does this correspond to?

3. **Gravitational redshift**
   (a) Explain how gravitational potential wells on the surface of last scattering cause temperature anisotropies.
   (b) Explain the difference between the Sachs-Wolfe and integrated Sachs-Wolfe effects.
   (c) Explain how the integrated Sachs-Wolfe effect would change if there was no dark energy in our Universe.

4. **Baryon acoustic oscillations**
   (a) By drawing a series of simple sketches, explain how the baryon acoustic oscillations arise.
   (b) Describe how Doppler shifts can give rise to temperature anisotropies. Use this to explain why the power spectrum does not fall to zero in between the baryon acoustic oscillation peaks of the CMB power spectrum.

5. **Angular power spectrum**
   (a) Sketch the power spectrum of the CMB as a function of spherical harmonic mode, ℓ. Label the Sachs-Wolfe plateau, baryon acoustic oscillation peaks, and damping tail.
   (b) How would your plot be affected if the surface of last scattering was further away from us (i.e. at a greater angular diameter distance)? (*Hint:* Assume that the comoving sizes of the acoustic peaks don't change.)

6. **Designing a CMB experiment**
   Consider a telescope consisting of a microwave receiver attached to a parabolic dish. The angular resolution of the telescope depends on the observation wavelength λ and the diameter of the dish D, so that the smallest angular feature that it can resolve is $\Delta\theta \approx \lambda/D$ (where $\Delta\theta$ is in radians).
   (a) What sized dish is needed to resolve features that are 1 degree across at an observing frequency of 70 GHz?
   (b) The Wien Law tells us that the peak of a blackbody spectrum can be found at a wavelength of $\lambda_{\rm peak} = b/T$, where $b = 2898\mu$m K and T is the blackbody temperature. Assuming that our experiment observes at this wavelength, what sized dish does it need to resolve the CMB power spectrum out to $\ell = 1000$?
   (c) The South Pole Telescope observes at frequencies 90 GHz and 150 GHz, and has a dish that is 10m in diameter. What is the highest ℓ-mode that it can observe in each frequency band?

Assessed question: First acoustic peak of the CMB power spectrum

(a) Briefly explain the physical origin of the first acoustic peak in the CMB power spectrum.
[6 marks]

(b) What is the approximate numerical value of the comoving sound horizon at last scattering, r_s? Give your answer in Mpc.
[2 marks]

(c) The first acoustic peak is observed at a spherical harmonic mode of $\ell \approx 220$. Use this to calculate the corresponding angular scale (in degrees), and the angular diameter distance to the CMB (in Mpc).
[5 marks]

(d) Consider a scenario in which the acoustic peak had been observed at $\ell = 200$ instead. Assuming that the redshift of decoupling and the size of the comoving sound horizon are unchanged, calculate how this would change the angular diameter distance to last scattering.
[5 marks]

(e) The comoving sound horizon can be calculated by integrating the sound speed with respect to time,

$$r_s(t) = \frac{1}{a(t)} \int_0^t c_s \, dt.$$

By first converting the expression above into an integral with respect to scale factor or redshift, calculate r_s at a redshift $z(t_{\rm LS}) = 1090$, assuming (i) a radiation-only cosmological model at $t < t_{\rm LS}$; (ii) a matter-only cosmological model at $t < t_{\rm LS}$. Compare your answers to part (b) above.

(Recall that $c_s^2 \approx c^2/3$ in our Universe. You may also assume that the expansion rate at $t_{\rm LS}$ is $H(t_{\rm LS}) \approx 1.384 \times 10^6\,{\rm km/s/Mpc}$, and $1\,{\rm km/s/Mpc} = 1.022 \times 10^{-12}\,{\rm yr}^{-1}$.)
[7 marks]