Week 5: Acceleration and the equation of state

This sheet doesn't need to be handed in for marking, but please bring it (with completed answers!) to next week's tutorial so we can go over the answers.

1. **Equation of state of radiation**
   Using only the conservation equation and the fact that $\rho_r \propto a^{-4}$, calculate the equation of state w for (non-interacting) radiation in a universe that contains matter, radiation, and a cosmological constant.

2. **Arbitrary equation of state**
   Consider a type of matter/energy that has an arbitrary equation of state that depends on scale factor, $w(a)$. We will approximate it using a Taylor expansion about $a = 1$, given by $w(a) = w_0 + w_a(1 - a)$, where w_0 and w_a are coefficients to be specified.

   Using only the conservation equation, find an expression for the energy density, $\rho(a)$, of this type of matter/energy as a function of scale factor, a.

3. **Deceleration parameter**
   The scale factor can be written as a Taylor expansion about $t = t_0$:

   $$a(t) \approx 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2 (t - t_0)^2 + \dots$$

   (a) Using the definition of the deceleration parameter from the notes,

   $$q(a) = -\left(\frac{a}{\dot{a}}\right)^2 \frac{\ddot{a}}{a},$$

   and the Taylor expansion above, show that $q(a) = q_0$ at $t = t_0$.

   (b) Use the same definition of $q(a)$ from part (a) above to show that

   $$q(a) = -\left(1 + \frac{\dot{H}}{H^2}\right).$$

4. **Marginally-accelerating universe**
   Consider a universe that is neither accelerating nor decelerating, i.e. $\ddot{a} = 0$, and that is filled with only a single kind of matter/energy with a constant equation of state parameter w.

   Find the value of w that is needed to produce this result.

Practice exam question: de Sitter universe

In the far future, our Universe is expected to asymptotically approach a de Sitter spacetime, which is spatially flat and has $\Omega_\Lambda = 1$.

(a) Write down the Friedmann and Raychaudhuri equations for a flat, Λ-only universe.
[4 marks]

(b) Find a solution for $a(t)$ in this universe.
[6 marks]

(c) Derive an expression the expansion rate, $H(a)$, in this universe.
[3 marks]

(d) By using the conservation equation, show that $w = -1$ in this universe.
[6 marks]

(e) Find an expression for the Hubble radius, $r_{\mathrm{HR}}(a)$, in this universe as a function of scale factor. Sketch a graph to show how it depends on a.
[6 marks]