Week 3: Friedmann equation

Please hand in the completed problems by **Wednesday 16th of October at 4pm**. *Please show your working and write neatly, staple all sheets together, and write your name and student number at the top of the first sheet.*

1. **Maths practice: ordinary differential equations**
   (a) Solve $dx/dt = \pm x^{-\frac{3}{2}}$ when $x(t_0) = 1$.
   (b) Solve $d^2y/dt^2 = \sqrt{\lambda}y$ to find $y(t)$.
   (c) Solve $p\,dp = \sqrt{p^2 + \kappa p^3}\,du$ to find $u(p)$.

2. **Unit conversions and density parameters**
   Take a universe with $H_0 = 70$ km/s/Mpc.
   (a) What is the critical density today in units of (i) kg m^{-3} and (ii) $M_\odot\,\mathrm{Mpc}^{-3}$?
   (b) Earth is approximately 8 kpc from the centre of the Milky Way. How many lightyears is that?
   (c) Imagine an object of length 3m is expanding with the Hubble flow. How much length will it gain after a year? What is the length gain as a fraction of its original length?
   (d) Repeat exercise (c) for an object of length 10 Mpc.

3. **Spatial curvature**
   Consider a universe with $H_0 = 80$ km/s/Mpc, $\Omega_\mathrm{m} = 0.8$, and $\Omega_\mathrm{k} = 0.2$.
   (a) Is this universe flat, open, or closed?
   (b) Sketch roughly how a triangle would appear in this space. Do the angles of the triangle add up to 180°?
   (c) Describe the ultimate fate of this universe, far into its future.
   (d) How many hydrogen atoms would you need to add or remove per m^3 of space at $t = t_0$ to make this universe flat?

4. **Matter-dominated universe** Consider a flat, expanding universe filled only with pressureless matter, and a Hubble expansion rate today of $H_0 = 67$ km/s/Mpc.
   (a) Write down the Friedmann equation for this universe.
   (b) What is the critical density at the present day in this universe, in units of $M_\odot\,\mathrm{Mpc}^{-3}$?
   (c) Solve for the scale factor, $a(t)$. What is the present age of this universe?

Assessed question: Applications of the Friedmann equation

(a) Write the Friedmann equation in terms of the fractional density parameters, Ω. Include appropriate scale factor-dependent terms for matter, radiation, curvature, and a cosmological constant.
[4 marks]

(b) For a fixed value of H_0, put the following (spatially-flat) universes in increasing order of age: pure matter-dominated; pure radiation-dominated; 99% matter and 1% radiation at t_0 = today.
[6 marks]

(c) What is the redshift of matter-radiation equality (when $\rho_m(t) = \rho_r(t)$) in the mixed matter + radiation universe from (b)?
[4 marks]

(d) Derive an expression for the deceleration parameter, $q(a) = -(1+\dot{H}/H^2)$, in a universe with both matter and radiation. (Make sure to simplify the final expression as much as you can.)
[7 marks]

(e) Use your expression to calculate $q(t_0)$ in a matter-only and a radiation-only universe respectively.
[4 marks]