Queen Mary
University of London

MSc Examination

Day 15th May 2014 14:30 – 17:00

ASTM109 Stellar Structure and Evolution Duration: 2.5 hours

YOU ARE NOT PERMITTED TO READ THE CONTENTS OF THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR.

Instructions:

Answer ALL question from Section A. Answer ONLY TWO questions from Section B. Section A carries 50 marks; each question in Section B carries 25 marks.

If you answer more questions than specified, only the first answers (up to the specified number) will be marked. Cross out any answers that you do not wish to be marked.

Calculators ARE permitted in this examination. The unauthorised use of material stored in pre-programmable memory constitutes an examination office. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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EXAM PAPERS MUST NOT BE REMOVED FROM THE EXAM ROOM.

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You may assume the following:
In all questions: M is the mass, $m(r)$ the mass interior to radius r, R is the radius, L the luminosity and $T_{\rm eff}$ the effective temperature of a star. P, ρ and T denote the pressure, density and temperature respectively. κ is the opacity per unit mass, ϵ the rate of energy production per unit mass and μ denotes the mean molecular weight. c_p, c_v are the specific heats at constant pressure and volume, $\gamma = c_p/c_v$ and $\mathcal{R}$ is the gas constant where $\mathcal{R} = \mu(c_p - c_v)$.
$L = 4\pi R^2 F_{Rad}$ and F_{Rad} is given by

$$F_{Rad} = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{dT}{dr}.$$

$c, G, \sigma = ac/4$ are respectively the velocity of light, the constant of gravity and the Stefan-Boltzmann radiation constant. X, Y, Z are the mass fractions respectively of hydrogen, helium and the heavier elements.
The central density ρ_c, central temperature T_c and central pressure P_c of a polytrope of index n are

$$\rho_c = a_n\frac{3M}{4\pi R^3}, \qquad T_c = b_n\frac{\mu GM}{\mathcal{R}R}, \qquad P_c = c_n\frac{GM^2}{R^4}.$$

The apparent magnitude $m_{\rm app}$, absolute magnitude $M_{\rm abs}$ and distance in parsecs d are related by $m_{\rm app} = M_{\rm abs} + 5\log d - 5$. The following rounded numerical values, all in S.I. Units may be assumed throughout the paper.

$$c = 3\times10^8, G = 7\times10^{-11}, \sigma = 6\times10^{-8}, M_\odot = 2\times10^{30}, R_\odot = 7\times10^8, L_\odot = 4\times10^{26}.$$

You may also assume that 1 year is 3×10^7seconds.

SECTION A Answer ALL questions in Section A

Question A1

A star S has a measured parallax of 0.13 arcsec, and an apparent magnitude of 0.03. Spectroscopic measurements show that the effective temperature of S is two times larger than that of the Sun.

a) What is the visible colour of the star S: is it redder or bluer than the Sun?

b) What is the distance to the star S (in parsecs)? What is its absolute magnitude?

c) The Sun has an absolute magnitude of 4.62. What is the apparent magnitude of a star identical to the Sun, when it is placed at the same distance as S?

[8 marks]

Question A2

a) Show that for a fully ionized gas consisting of atomic hydrogen and helium only, the mean molecular weight μ is given by

$$\mu = \frac{4}{8-5Y}.$$

b) For a particular homogeneous star, $Y = 0.25$ while for a second homogeneous star that is more evolved, Y has increased to 0.35. Both stars have the same polytropic index . The central temperature is the same in both stars and may be assumed to be as given in the rubric. If the second star has twice the mass of the first star, calculate the ratio of their radii.

[9 marks]

Question A3

The Lane-Emden equation is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n,$$

where constant n is the polytropic index. In polytropic stars, the radial profiles of density and pressure are governed by the solution of the Lane-Emden equation $\theta(\xi)$ as

$$\rho(r) = \rho_c\theta^n(\xi), \quad P(r) = P_c\theta^{n+1}(\xi),$$

where ρ_c and P_c are central values of density and pressure, and $r = \alpha\xi$ with some constant α. Consider a polytropic star with $n = 1$.

a) Show by direct substitution that the solution to the Lane-Emden equation is

$$\theta(\xi) = \frac{\sin\xi}{\xi}.$$

b) Deduce the value of ξ at the surface of the star.

c) Find the density (in terms of the central density) and temperature (in terms of the central temperature) at a distance from the centre of 1/3 of the radius. You may assume that the equation of state is that of an ideal gas, and that the mean molecular weight is uniform throughout the star.

[8 marks]

Question A4

You are given that the gravitational binding energy Ω of a star of polytropic index n is

$$\Omega = -\frac{3GM^2}{(5-n)R},$$

and it is related to the internal energy U by the virial theorem

$$2U + \Omega = 0.$$

a) Consider a star with a polytropic index $n = 3$ and no nuclear energy sources. The opacity is assumed to be of the Kramers type so that $L \propto M^{5.5}R^{-0.5}$. Show that such a star will evolve along a line in the HR diagram with a slope of 0.8.

b) Show that t, the time taken by such a star to evolve from a very large radius to some smaller radius R_0, is given by

$$t \propto M^{-3.5}R_0^{-0.5}.$$

[9 marks]

Question A5

Consider a group of homogeneous stars. Each star in the group has the same chemical composition, is homogeneous, and is composed of an ideal gas. Energy generation is by the p-p chain, with $\epsilon = \epsilon_0 \rho T^4$. All the energy is carried by radiation and the opacity is given by $\kappa = \kappa_0 \rho T^{-3.5}$.

a) Show that

$$M \propto R^{13}.$$

b) Also show that

$$L \propto M^{71/13}.$$

c) Obtain the slope of the line in an H-R diagram ($\log L$ versus $\log T_{\mathrm{eff}}$) that these stars lie on.

[9 marks]

Question A6

In the stellar core where pressure P is dominated by the pressure of the degenerate relativistic electrons, the pressure and density are related by a polytropic law

$$P = K\rho^{\frac{4}{3}},$$

where K is some constant.

a) Show that there is only one possible mass for a stellar core where pressure is dominated by the pressure of the degenerate relativistic electrons.

b) Show that when the pressure and the internal energy in the stellar core are dominated by those of the degenerate relativistic electrons,

$$U + \Omega = 0,$$

where U is thermal energy, and Ω is gravitational binding energy of the core. Discuss briefly the physical meaning of this result. You may assume without proof that the internal energy density u is $u = 3P$, and

$$\Omega = -3 \int_V P \, dv,$$

where V is the spherical volume occupied by the core.

[7 marks]

SECTION B Answer TWO questions from Section B

Question B1

a) By considering the forces acting on a volume element, show that for a spherically symmetric star to be in hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2}.$$

[6 marks]

b) Consider a hypothetical star of mass M and radius R, with uniform density distribution (ρ=const). Find the expressions for $m(r)$ and $P(r)$.

[6 marks]

c) Material in the star satisfies the ideal-gas equation of state $P = \frac{\mathcal{R}}{\mu}\rho T$. Deduce whether the star is convectively stable or unstable.

[3 marks]

d) Obtain an expression for the gravitational binding energy Ω of the star in terms of M and R.

[5 marks]

e) For an ideal gas of classical particles, the internal energy density u is $u = \frac{3}{2}P$ (you can use this relation without a proof). Obtain an expression for the total internal energy U of the star in terms of M and R. Verify that $\Omega = -2U$.

[5 marks]

Question B2

a) The interaction of photons with atoms is described in terms of a cross section σ_{R}, which is defined such that

$$n\lambda_{\mathsf{ph}}\sigma_{\mathsf{R}} = 1,$$

where n is the number of atoms per unit volume, and λ_{ph} is the mean free path of a photon. Using geometrical arguments, explain the origin of this definition.

[3 marks]

b) The opacity κ is defined as

$$\kappa = \frac{1}{\rho\lambda_{\mathsf{ph}}}.$$

Show that κ is the total cross section per unit mass.

[3 marks]

c) The optical depth, τ, in a stellar atmosphere is defined as

$$\tau = \int_r^\infty \kappa\rho\, dr.$$

Starting from the equation for radiative flux, F, given in the rubric, show that in the atmosphere of a star

$$T^4 = \frac{3F}{ac}\left(\tau + B\right),$$

where B is a constant of integration, which you may assume without proof to be $B = 2/3$. Using $F = \sigma T_{\text{eff}}^4$ as a definition of the effective temperature T_{eff}, show that

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \left(\tau + \frac{2}{3}\right).$$

[8 marks]

d) Assume now that this $T - \tau$ relation is valid everywhere in the radiative atmosphere, including optically thin layers. Explain the possible weaknesses of this assumption. Assume further that the atmosphere is an ideal gas and in hydrostatic equilibrium and that the mass and thickness of the atmosphere are both negligible compared to the mass and radius of the star. Given that $P = 0$ at $\tau = 0$ and that the opacity is given by $\kappa = \kappa_0 \rho T^5$ with some constant κ_0, show that

$$P^2 = P_0^2 \ln\left(1 + \frac{3}{2}\tau\right),$$

where P_0 is another constant which you do not need to specify.

[11 marks]

Question B3

a) The adiabatic exponent γ is defined as

$$\gamma = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S,$$

where subscript S indicates that the partial derivative is taken at constant entropy, i.e. without any heat exchange. Assuming that the equation of state is that of an ideal gas, $P = \frac{\mathcal{R}}{\mu}\rho T$, show that

$$\left(\frac{\partial \ln T}{\partial \ln P}\right)_S = \frac{\gamma - 1}{\gamma}.$$

[3 marks]

b) Derive the Schwarzschild condition for the onset of convection in an ideal gas, namely

$$\frac{d \ln T}{d \ln P} > \frac{\gamma - 1}{\gamma}.$$

[11 marks]

c) Assume that the temperature- and pressure profiles in the radiative stellar atmosphere are given by the relations

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \left(\tau + \frac{2}{3} \right),$$

$$P^2 = P_0^2 \ln \left(1 + \frac{3}{2} \tau \right),$$

where τ is an optical depth, and P_0 is some constant. Assume further that $\gamma = 5/3$. Show that the convection sets in at a level where

$$\tau = \frac{2}{3} \left[\exp \left(\frac{4}{5} \right) - 1 \right].$$

[11 marks]

Question B4

According to Pauli's exclusion principle, at most two electrons can occupy a given energy state, and each particular energy state occupies volume h^3 in the 6-dimensional space of coordinates and momenta, where h is the Planck constant.

a) In a degenerate gas all the electron states are filled up to a threshold momentum p_F and none above. Show that the number density of electrons for which momentum p is in the interval $(p, p + dp)$ is

$$n_e(p)\, dp = \frac{8 \pi p^2}{h^3}\, dp$$

when $p \leq p_F$. Show that the total (i.e., integrated over all the possible momenta) electron number density is

$$n_e = \frac{8\pi}{3h^3} p_F^3.$$

[7 marks]

b) Show that when the electrons are moving with speeds small compared to the speed of light, the energy density of the degenerate electrons is

$$u_e = \frac{4\pi}{5h^3 m_e} p_F^5,$$

and the pressure of the degenerate electrons is

$$P_e = \frac{8\pi}{15h^3 m_e} p_F^5.$$

You may use the general expression $E = \frac{1}{2}\frac{p^2}{m}$ for the energy E of a single particle with momentum p and mass m, and the general relation $P = \frac{2}{3}u$ between pressure P and energy density u in a gas composed of classical particles.

[7 marks]

c) Show that in a completely ionized gas, the electron number density, n_e, is approximately

$$n_e \simeq \frac{\rho(1+X)}{2m_{\mathsf{H}}},$$

where X is mass fraction of hydrogen, and m_{H} is the mass of the hydrogen atom.

[4 marks]

d) Show that the analysis of part (b) of this question, which neglects the relativistic effects, is only applicable when

$$\rho \ll m_{\mathsf{H}} \left(\frac{m_e c}{h}\right)^3,$$

where c is speed of light.

[7 marks]

End of Paper